1 Homework #8

Consider the Hamiltonian system for the harmonic oscillator. The Hamiltonian is

$$H(x,\pi) = \frac{\pi^2}{2m} + \frac{kx^2}{2}$$

The corresponding system of differential equations is

$$\frac{dx}{dt} = \frac{\partial H}{\partial \pi} = \frac{x}{m} \tag{1}$$

$$\frac{d\pi}{dt} = -\frac{\partial H}{\partial x} = -kx \tag{2}$$

$$\tag{3}$$

1. $x^* = x + \pi, \pi^* = \pi$.

$$\frac{dx^*}{dt} = \frac{dx}{dt} + \frac{d\pi}{dt} = \frac{\pi}{m} - kx = \frac{\pi^*}{m} - k(x^* - \pi^*) \tag{4}$$

$$\frac{d\pi^*}{dt} = \frac{d\pi}{dt} = -kx = k(\pi^* - x^*) \tag{5}$$

2.

$$H^*(x^*,\pi^*) = \frac{(\pi^*)^2}{2m} + \frac{k(x^* - \pi^*)^2}{2}$$

$$\frac{\partial H^*}{\partial \pi^*} = \frac{\pi^*}{m} - k(x^* - \pi^*) = \frac{dx^*}{dt} \tag{6}$$

$$\frac{\partial H^*}{\partial x^*} = k(x^* - \pi^*) = -\frac{d\pi^*}{dt} \tag{7}$$

Therefore, we get the same set of differential equations as in (1), so the transformation preserves the form of the Hamiltonian equation.

3. $x^* = e^x, \pi^* = \pi$.

$$\frac{dx^*}{dt} = e^x \frac{dx}{dt} = x^* \frac{\pi}{m} = \frac{x^* \pi^*}{m} \tag{8}$$

$$\frac{d\pi^*}{dt} = \frac{d\pi}{dt} = -kx = -k\ln(x^*) \tag{9}$$

4.
$$H^*(x^*,\pi^*) = \frac{(\pi^*)^2}{2m} + \frac{k(\ln(x^*))^2}{2}$$

$$\frac{\partial H^*}{\partial \pi^*} = \frac{\pi^*}{m} = \frac{dx^*}{dt} \tag{10}$$
$$\frac{\partial H^*}{\partial x^*} = \frac{k\ln(x^*)}{x^*} = -\frac{d\pi^*}{dt} \tag{11}$$

which are different than the equations in (3). Therefore the transformation is not canonical.

5. $x^* = e^x$, we can set $\tilde{G} = \tilde{G}(x, \pi^*)$.

By case 2 in class, we have

$$x^* = -\tilde{G}_{\pi^*} = e^x \tag{12}$$
$$\tilde{G} = -e^x\pi^* \tag{13}$$
$$\pi = -\tilde{G}_x = e^x\pi^* \tag{14}$$
$$\pi^* = \pi e^{-x} \tag{15}$$

Thus, with $x^* = e^x$, $\pi^* = \pi e^{-x}$ is a canonical transformation.

6. For G we have

$$dG = \sum_{i=1}^{m} G_{u_i} du_i + \sum_{i=1}^{m} G_{u_i^*} du_i^* + G_x dx \tag{16}$$

But since $G_x = 0 \Longrightarrow H^* = H$. Thus,

$$dG = \sum_{i=1}^{m} \pi_i^* du_i^* - \sum_{i=1}^{m} \pi_i du_i \tag{17}$$
$$= \sum_{i=1}^{m} \pi_i^* du_i^* - \sum_{i=1}^{m} \pi_i du_i + (H^* - H) \tag{18}$$

which gives us the transformation is canonical.

7. $u^* = \sqrt{u}\cos(2\pi), \pi^* = \sqrt{u}\sin(2\pi)$.

$$\pi^* du^* - \pi du = (\frac{1}{4}\sin(4\pi) - \pi)du - 2u\sin^2(2\pi)d\pi \tag{19}$$
$$:= F_1 du + F_2 d\pi \tag{20}$$

We can check that $\frac{\partial F_1}{\partial \pi} = \frac{\partial F_2}{\partial u}$, therefore this is an exact differential and we can solve for G such that $dG = \pi^* du^* - \pi du$. Specifically,

$$G = \frac{u}{4}\sin(4\pi) - u\pi.$$

Thus, by problem (6), this is a canonical transformation.