1 Homework #9

1. Consider a particle which is attracted to the origin with a force that depends only on the distance from the origin.

   (a) Show that the Lagrangian for this motion has the form

   $$L(X, V) = \frac{1}{2}m|V|^2 - f(|X|)$$

   where X is the position, V is the velocity, and f is some function.

   Solution There is an error in the statement of this problem, so reasonable discussions of kinetic and potential energies will constitute correct answers.

   (b) Examine the quantities energy, (linear) momentum, and angular momentum. Which are conserved and which are not?

   Solution Since

   $$H = L - V \cdot L_V = -\frac{1}{2}m|V|^2 - f(|X|) = -E,$$

   the energy is constant in time, so energy is conserved.

   Linear momentum is not conserved, because $\frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i$ is not constant.

   Angular momentum is conserved, which can be shown either by direct computation as per the transformation on page 87 of the text, or by noting that velocity and distance to the origin are independent of rotation.

2. Consider the functional $\mathcal{F}(u) = \int_a^b xu'(x)^2 dx$.

   (a) Show $\mathcal{F}$ is invariant under $x^* = x, u^* = u + \epsilon$.

   Solution Find that

   $$\frac{du^*}{dx^*} = \frac{d(u+\epsilon)}{dx} = \frac{du}{dx}$$

   $$a^* = a, b^* = b,$$

   so

   $$\mathcal{F}(u^*, I^*) = \int_{a^*}^{b^*} x^* \left(\frac{du^*}{dx^*}\right)^2 dx^* = \int_a^b xu'(x)^2 dx = \mathcal{F}(u).$$

   Therefore, $\mathcal{F}$ is invariant under translation in u.

   (b) What is the conserved quantity that is predicted by Noether's Theorem and part (a)?

   Solution Find that $\mu(x, u, p) = 0$, $\phi(x, u, p) = 1$, and $F = xp^2$. By Noether's Theorem, $F_p = 2xp = 2xu'$ is conserved.

(c) Use part (b) to find the extremals of $\mathcal{F}$.

Solution Calculating,

$$\begin{aligned} xu' &= C_1 \\ u' &= \frac{C_1}{x}, \text{ so} \\ u &= C_1 \ln(x) + C_2 \text{ is the extremal.} \end{aligned}$$

(d) Show that $\mathcal{F}$ is invariant under the transformations $x^* = (1+\varepsilon)x$ and $u^* = u$.

Solution Find that

$$\frac{du^*}{dx^*} = \frac{du}{dx^*} = \frac{1}{1+\epsilon}\frac{du}{dx},$$

$$a^* = (1+\epsilon)a, \;\; b^* = (1+\epsilon)b,$$

so that

$$\mathcal{F}(u^*, I^*) = \int_{a^*}^{b^*} x^* \left(\frac{du^*}{dx^*}\right)^2 dx^* = \int_{a^*}^{b^*} x^* \frac{1}{(1+\epsilon)^2}\left(\frac{du}{dx}\right)^2 dx^*.$$

Changing variables, we get

$$\mathcal{F}(u^*, I^*) = \int_a^b x(1+\epsilon)\frac{1}{(1+\epsilon)^2}\left(\frac{du}{dx}\right)^2 (1+\epsilon)dx = \int_a^b xu'(x)^2 dx = \mathcal{F}(u).$$

Therefore, $\mathcal{F}$ is invariant.

(e) What is the conserved quantity that is predicted by Noether's Theorem and part (d)?

Solution Find that $\mu(x,u,p) = x$, $\phi(x,u,p) = 0$, and $F = xp^2$, so by Noether's Theorem, $x(xp^2 - p2xp) = -x^2p^2$ is conserved. This can be restated as xu' being conserved.

3. In exercise 3 of homework 7, you found the extremals of the functional

$$\mathcal{F}(u) = \frac{1}{2}\int_a^b [u' + ku]^2 dx.$$

(a) Show that $\mathcal{F}$ is invariant under the transformations

$$x^* = x + \varepsilon, \text{ and } u^* = u + \varepsilon\alpha e^{-kx}$$

for any constant α.

Solution We have

$$\frac{du^*}{dx^*} = \frac{du^*}{dx} = \frac{du}{dx} - \epsilon\alpha k e^{-kx}$$

$$a^* = a + \epsilon,\ b^* = b + \epsilon,$$

so

$$\mathcal{F}(u^*, I^*) = \frac{1}{2}\int_{a^*}^{b*}\left[\frac{du^*}{dx^*} + ku^*\right]^2 dx^* = \frac{1}{2}\int_{a^*}^{b*}\left[\frac{du}{dx} - k\epsilon\alpha e^{-kx} + ku + k\epsilon\alpha e^{-kx}\right]^2 dx^*.$$

After canceling terms, and changing variables,

$$\mathcal{F}(u^*, I^*) = \frac{1}{2}\int_a^b \left[\frac{du}{dx} + ku\right]^2 dx = \mathcal{F}(u, I).$$

Therefore, $\mathcal{F}$ is invariant.

(b) What is the conserved quantity that is predicted by Noether's Theorem and part (a)?
Solution With $\mu = 1, \phi = \alpha e^{-kx}, F = (p + ku)^2$, by Noether's Theorem,

$$(p+ku)^2 - 2p(p+ku) + \alpha e^{-kx} 2(p+ku) = k^2u^2 - p^2 + 2\alpha e^{-kx}(p+ku)$$

is conserved.

(c) Use the result of part (b) for $\alpha = 0$ and $\alpha = 1$ to find the extremals.
Solution When $\alpha = 0$,

$$k^2u^2 - p^2 = C_1$$

and when $\alpha = 1$,

$$k^2u^2 - p^2 + 2e^{-kx}(p + ku) = C_2$$

Subtracting these two expressions, we obtain

$$2e^{-kx}(p + ku) = C_3$$

or

$$u' + ku = C_3 e^{kx}$$

Solving this equation, we get

$$u(x) = Ae^{kx} + Be^{-kx}$$

for constants A, B.

4. Write and solve the Hamilton-Jacobi equation corresponding to the functional

$$J[y] = \int_{x_0}^{x_1} y'^2 \, dx,$$

and use the result to determine the extremals of $J[y]$.

Solution The generalized momentum is $\pi = \frac{\partial F}{\partial y'} = 2y'$. The Hamiltonian is $H(x, y, \pi) = \frac{1}{4}\pi^2$. The Hamilton-Jacobi equation is then

$$\frac{\partial S}{\partial x} + \frac{1}{4}\left(\frac{\partial S}{\partial y}\right) = 0.$$

Assuming a separable solution of the form

$$S(x, y) = f(x) + g(y),$$

the Hamilton-Jacobi equation becomes

$$4f'(x) + g'(y)^2 = 0.$$

Solving the above yields that the extremals of J are straight lines.

5. Write and solve the Hamilton-Jacobi equation corresponding to the functional

$$J[y] = \int_{x_0}^{x_1} f(y)\sqrt{1 + y'^2} \, dx,$$

and use the result to find the extremals of $J[y]$.

Solution The generalized momentum is $\pi = \frac{\partial F}{\partial y'} = \frac{f(y)y'}{\sqrt{1+y'^2}}$. The Hamiltonian is $H(x, y, \pi) = \pi y' - f(y)\sqrt{1 + y'^2}$. With $y' = \frac{\pi}{\sqrt{f(y)^2 - \pi^2}}$, find that

$$H(x, y, \pi) = \frac{\pi^2}{\sqrt{f(y)^2 - \pi^2}} - f(y)\sqrt{1 + \frac{\pi^2}{f(y)^2 - \pi^2}}.$$

The Hamilton-Jacobi equation simplifies to

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 = f(y)^2.$$

Assuming a separable solution of the form

$$S(x, y) = u(x) + v(y),$$

the Hamilton-Jacobi equation yields $u'(x)^2 = \alpha^2$ and $v'(y)^2 = \sqrt{f(y)^2 - \alpha^2}$. Then $u(x) = \alpha x$ and $v(y) = \int_{y(x_0)}^{y} \sqrt{f(\eta)^2 - \alpha^2} \, d\eta + B$.

Taking the derivative of S with respect to α yields the result from the book that extremals satisfy

$$x - \alpha \int_{y(x_0)}^{y} \frac{d\eta}{\sqrt{f(\eta)^2 - \alpha^2}}$$

constant.