**Math 410**
**Final Exam**

April 23, 2008

**Instructions:**

This is an open book, open notes, take home exam. It is due by 5:00 PM in HB 450 on Wednesday, April 30 if you are a degree candidate, or on Monday, May 6 if you are not.

In concurrence with University regulations there is a five hour time limit on the exam.

Over the course of the semester we learned several different ways to solve problems in the calculus of variations. Except where specific instructions are given, no specific method is required. Just solve the problems.

1. Consider the functional

$$\mathcal{F}(u) = \int_a^b \left[u(x)^2 - u'(x)^2 - 2u(x)\sin(2x)\right]\, dx.$$

   (a) Find all extremals.
   (b) What is the extremal that satisfies $u(0) = 0$ and $u(\pi/2) = 1$?

2. Suppose that $F = F(p)$ is a function of one variable, and $F'$ is non-constant. Find all extremals of the functional

$$\mathcal{F}(u) = \int_a^b F(u'(x))\, dx.$$

3. Consider a function $u$ defined on $[0, 1]$ which satisfies

$$u(0) = u(1) = 0, \quad \text{and} \quad u(x) \geq 0.$$

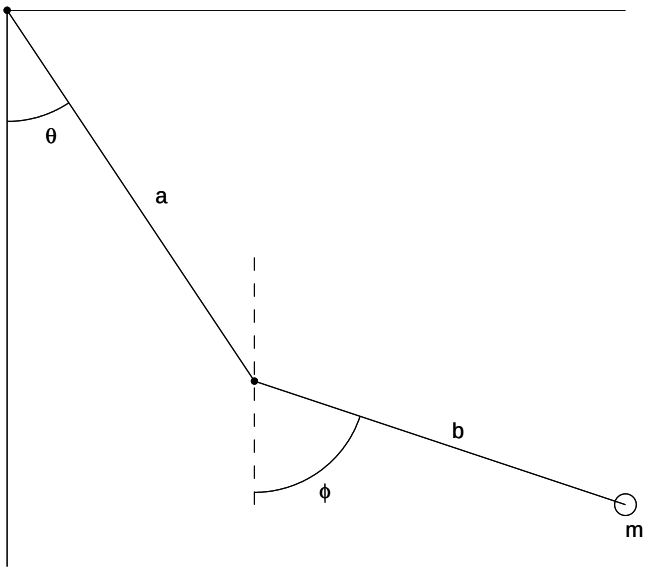
   When the graph of $u$ is rotated about the $x$-axis we get a surface of revolution, which is the boundary of a volume of revolution. Suppose that $V$ is a given quantity. Find the function $u$ which leads to a surface with minimum area while bounding a volume with volume $V$.

4. Consider the functional

$$\mathcal{F}(u) = \int_a^b \left[au'(x)^2 + 2bu(x)u'(x) + cu(x)^2\right]\, dx,$$

   where $a$, $b$, and $c$ are constants.
   (a) Find the extremals.
   (b) Can $\mathcal{F}$ have any broken extremals?

5. Consider the articulated pendulum in the figure below. (*Hint:* The algebra in this problem will be somewhat easier if you remember the formulas for the sum and differences of angles.)




An articulated pendulum.

(a) Show that in terms of the angles $\theta$ and $\phi$, the kinetic energy of the system is

$$T = \frac{m}{2}\left[\left(a\dot{\theta}\sin\theta + b\dot{\phi}\sin\phi\right)^2 + \left(a\dot{\theta}\cos\theta + b\dot{\phi}\cos\phi\right)^2\right].$$

(b) Show that in terms of the angles $\theta$ and $\phi$, the potential energy of the system is

$$V = -mg\left(a\cos\theta + b\cos\phi\right).$$

(c) Derive the differential equations of motion in terms of the angles $\theta$ and $\phi$.

6. Consider the Lagrangian

$$F(u,p) = \frac{1}{2}|p|^2|u|^2 - \frac{1}{|u|^2},$$

where $u = (u^1, u^2)$ has values in $\mathbf{R}^2$.

(a) Find the canonical variables and the Hamiltonian.
(b) What is the Hamiltonian system?
(c) Solve the corresponding Hamilton-Jacobi equation.
(d) Solve for the motion $u(t)$ in terms of unevaluated integrals.

7. We will use $\mathcal{F}(u) = \int_a^b F(u(x), u'(x))\,dx$ where $F$ is defined in Problem 6.

(a) Show that $\mathcal{F}$ is invariant under rotations in the variable $u$.
(b) What is the quantity that is constant along solutions predicted by Noether's theorem? as a result of part a)?