

Math 410
Final Exam

April 16, 2009

Instructions:

This is an open book, open notes, take home exam. It is due by 5:00 PM under the door of HB 450 on Wednesday, April 29.

In concurrence with University regulations there is a five hour time limit on the exam. It is permissible to take a break during the five hours, but you may not consult anybody or anything during that break.

Over the course of the semester we learned several different ways to solve problems in the calculus of variations. Except where specific instructions are given, no particular method is required. Just solve the problems.

1. Consider the functional

$$\mathcal{F}(u) = \int_a^b [u'(x)^2 - u(x)^2 + 2u(x)e^x] dx.$$

- (a) Find all extremals.
 - (b) Is the extremal that satisfies $u(0) = 0$ and $u(\pi/2) = 0$ a weak minimizer?
 - (c) Is the extremal that satisfies $u(0) = 0$ and $u(3\pi/2) = 0$ a weak minimizer?
 - (d) Can the functional \mathcal{F} have any broken extremals?
2. Consider the functional

$$\mathcal{F}(u) = \int_{-1}^1 \frac{x^2}{2} \left[u'(x)^2 - \frac{1}{3}u(x)^6 \right] dx.$$

- (a) Show that $\mathcal{F}(u)$ is invariant under the change of coordinates

$$y = (1 + \epsilon)x, \quad v = \frac{u}{\sqrt{1 + \epsilon}},$$

for all ϵ .

- (b) What is the quantity that is constant along extremals according to Noether's theorem?
3. A mass is constrained to move on the surface of a sphere of radius R under the force of gravity. If z is the vertical direction, this means that the potential is $U = mgz$, where m is the mass and g is a constant. (This is a spherical version of the pendulum.) Use spherical coordinates with the polar angle θ measured from the south pole, so that

$$\begin{aligned}x &= R \sin \theta \cos \phi \\y &= R \sin \theta \sin \phi \\z &= -R \cos \theta\end{aligned}$$

- (a) Show that the Lagrangian in these coordinates is given by

$$L(\theta, \phi, \dot{\theta}, \dot{\phi}) = \frac{mR^2}{2} [\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta] + mgR \cos \theta.$$

- (b) Find the Lagrangian differential equations of motion for the mass.
 (c) Find the generalized momenta π_θ and π_ϕ , and express the Hamiltonian in terms of them.
 (d) Find the differential equations of motion in Hamiltonian form.
 (e) Solve the Hamilton-Jacobi equation.
 (f) Solve for $\theta(t)$ and $\phi(t)$ in terms of unevaluated integrals.
4. Consider the functional $\mathcal{F}(u) = \int_0^b u'(x)^2 dx$.

- (a) Find all extremals for $\mathcal{F}(u)$ subject to the constraint

$$\int_0^b u(x)^2 dx = 1,$$

and the boundary conditions $u(0) = u(b) = 0$.

- (b) Among all of the extremals you found in part (a), which could be a minimizer for $\mathcal{F}(u)$ subject to the constraints in part (a)? A proof is not necessary.
 (c) Assuming that your answer in part (b) is really the minimizer of the constrained problem in part (a), prove the following version of Poincaré's inequality:

$$\int_0^b v(x)^2 dx \leq \frac{b^2}{\pi^2} \int_0^b v'(x)^2 dx \quad \text{for all } v \in C_0^1(0, b).$$

(*Hint:* If $v \in C_0^1(0, b)$ is not the zero function, can you find a constant c such that $u = cv$ satisfies the constraint in part (a)?)