Math 410

Final Exam

April 16, 2009

Instructions:

This is an open book, open notes, take home exam. It is due by 5:00 PM under the door of HB 450 on Wednesday, April 29.

In concurrence with University regulations there is a five hour time limit on the exam. It is permissable to take a break during the five hours, but you may not consult anybody or anything during that break.

Over the course of the semester we learned several different ways to solve problems in the calculus of variations. Except where specific instructions are given, no particular method is required. Just solve the problems.

1. Consider the functional

$$\mathcal{F}(u) = \int_a^b \left[u'(x)^2 - u(x)^2 + 2u(x)e^x \right] \, dx.$$

(a) Find all extremals.

Answer: The Lagrangian is $F(x, u, p) = p^2 - u^2 + 2ue^x$. We have $F_u = 2(e^x - u)$ and $F_p = 2p$, so the Euler Lagrange equation reduces to $u'' = -u + e^x$. The solutions are

$$u(x) = \frac{1}{2}e^x + A\cos x + B\sin x,$$

where A and B are arbitrary constants.

(b) Is the extremal that satisfies $u(0) = 0$ and $u(\pi/2) = 0$ a weak minimizer?

Answer: The intial conditions imply that $0 = u(0) = 1/2 + A$, or $A = -1/2$, and $0 = u(\pi/2) = e^{\pi/2}/2 + B$, or $B = -e^{\pi/2}/2$. Hence

$$u(x) = \frac{1}{2}\left(e^x - \cos x - e^{\pi/2}\sin x\right).$$

We have

- u is an extremal.
- $F_{pp} = 2$ is positive, so the strict Legendre condition is satisfied.
- Since $F_{up} = 0$ and $F_{uu} = -2$, the Jacobi equation reduces to $v'' + v = 0$. Hence the Jacobi fields have the form $v(x) = A\sin x$ for $A \neq 0$. The smallest point conjugate to 0 is $x = \pi$. Consequently there ae no conjugate points in $(0, \pi/2]$, so the strict Jacobi condition is satisfied.

Hence u is a strict weak minimizer.

(c) Is the extremal that satisfies $u(0) = 0$ and $u(3\pi/2) = 0$ a weak minimizer?

Answer: From what we discovered in part (b) we see that π is a conjugate point which is in the interval $(0, 3\pi/2)$. The Jacobi condition is not satisfied, so the extremal is not a weak minimizer.

(d) Can the functional $\mathcal{F}$ have any broken extremals?

Answer: Since $F_{pp} = 2 > 0$, every piecewise continuously differentiable weak extremal is twice continously differentiable. Hence there can be no broken extremals.

2. Consider the functional

$$\mathcal{F}(u) = \int_{-1}^{1} \frac{x^2}{2} \left[u'(x)^2 - \frac{1}{3} u(x)^6 \right] dx.$$

(a) Show that $\mathcal{F}(u)$ is invariant under the change of coordinates

$$y = (1+\epsilon)x, \quad v = \frac{u}{\sqrt{1+\epsilon}},$$

for all ϵ.

Answer: We have $x = \frac{y}{1+\epsilon}$, so

$$v(y) = \frac{1}{\sqrt{1+\epsilon}} u\left(\frac{y}{1+\epsilon}\right), \quad \text{and}$$
$$v'(y) = (1+\epsilon)^{-3/2} u'\left(\frac{y}{1+\epsilon}\right).$$

Hence

$$\begin{aligned}\mathcal{F}(v) &= \int_{-(1+\epsilon)}^{1+\epsilon} \frac{y^2}{2} \left[v'(y)^2 - \frac{1}{3} v(y)^6 \right] dy \\ &= \int_{-(1+\epsilon)}^{1+\epsilon} \frac{y^2}{2} \left[(1+\epsilon)^{-3} u'\left(\frac{y}{1+\epsilon}\right)^2 - \frac{1}{3}(1+\epsilon)^{-3} u\left(\frac{y}{1+\epsilon}\right)^6 \right] dy\end{aligned}$$

Substituting $y = (1+\epsilon)x$, this becomes

$$\begin{aligned}\mathcal{F}(v) &= \int_{-1}^{1} \frac{(1+\epsilon)^2 x^2}{2} \left[\frac{u'(x)^2 - u(x)^6/3}{(1+\epsilon)^3} \right] (1+\epsilon)\, dx \\ &= \int_{-1}^{1} \frac{x^2}{2} \left[u'(x)^2 - \frac{1}{3} u(x)^6 \right] dx \\ &= \mathcal{F}(u).\end{aligned}$$

(b) What is the quantity that is constant along extremals according to Noether's theorem?

Answer: The Lagrangian is

$$F(x,u,p) = \frac{x^2}{2}\left[p^2 - \frac{u^6}{3}\right],$$

so

$$\pi = F_p = x^2 p \quad \text{and} \quad H = pF_p - F = \frac{x^2}{2}\left[p^2 + \frac{1}{3}u^6\right].$$

The conserved quantity is $\mu H - \phi\pi$, where

$$\mu = \left.\frac{dy}{d\epsilon}\right|_{\epsilon=0} = x \quad \text{and} \quad \phi = \left.\frac{dv}{d\epsilon}\right|_{\epsilon=0} = -\frac{u}{2}.$$

This comes to

$$x^3\left[p^2 + \frac{1}{3}u^6\right] + x^2 up, \quad \text{or}$$

$$x^3\left[u'(x)^2 + \frac{1}{3}u(x)^6\right] + x^2 u(x)u'(x).$$

3. A mass is constrained to move on the surface of a sphere of radius R under the force of gravity. If z is the vertical direction, this means that the potential is $U = mgz$, where m is the mass and g is a constant. (This is a spherical version of the pendulum.) Use spherical coordinates with the polar angle θ measured from the south pole, so that

$$x = R\sin\theta\cos\phi$$
$$y = R\sin\theta\sin\phi$$
$$z = -R\cos\theta$$

(a) Show that the Lagrangian in these coordinates is given by

$$L(\theta,\phi,\dot\theta,\dot\phi) = \frac{mR^2}{2}\left[\dot\theta^2 + \dot\phi^2\sin^2\theta\right] + mgR\cos\theta.$$

Answer: Differentiating we get

$$\dot x = R\left[\dot\theta\cos\theta\cos\phi - \dot\phi\sin\theta\sin\phi\right]$$
$$\dot y = R\left[\dot\theta\cos\theta\sin\phi + \dot\phi\sin\theta\cos\phi\right]$$
$$\dot z = R\dot\theta\sin\theta.$$

The kinetic energy is

$$T = \frac{m}{2}|v|^2 = \frac{m}{2}(\dot x^2 + \dot y^2 + \dot z^2)$$
$$= \frac{mR^2}{2}\left[\dot\theta^2 + \dot\phi^2\sin^2\theta\right].$$

The Lagrangian is

$$L = T - U = \frac{mR^2}{2}\left[\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right] + mgR\cos\theta,$$

and the total energy is

$$H = T + U = \frac{mR^2}{2}\left[\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right] - mgR\cos\theta.$$

(b) Find the Lagrangian differential equations of motion for the mass.

Answer: There are two Euler Lagrange equation. First, $L_\theta = \frac{d}{dt}L_{\dot{\theta}}$ becomes

$$R\ddot{\theta} = R\dot{\phi}^2 \sin\theta\cos\theta - g\sin\theta.$$

The second, $L_\phi = \frac{d}{dt}L_{\dot{\phi}}$, simplifies because $L\dot{\phi} = 0$ to L_ϕ is a constant, or

$$\dot{\phi}\sin^2\theta = C.$$

(c) Find the generalized momenta π_θ and π_ϕ, and express the Hamiltonian in terms of them.

Answer: The generalized momenta are

$$\pi_\theta = L_{\dot{\theta}} = mR^2\dot{\theta} \quad \text{and} \quad \pi_\phi = L_{\dot{\phi}} = mR^2\dot{\phi}\sin^2\theta.$$

The Hamiltonian in terms of the momenta is

$$H = \frac{\pi_\theta^2}{2mR^2} + \frac{\pi_\phi^2}{2mR^2\sin^2\theta} - mgR\cos\theta.$$

(d) Find the differential equations of motion in Hamiltonian form.

Answer: The Hamiltonian equations are

$$\begin{aligned}
\theta' &= H_{\pi_\theta} = \frac{\pi_\theta}{mR^2} \\
\phi' &= H_{\pi_\phi} = \frac{\pi_\phi}{mR^2\sin^2\theta} \\
\pi_\theta' &= -H_\theta = \frac{\pi_\phi^2\cos\theta}{mR^2\sin^3\theta} + mgR\sin\theta \\
\pi_\phi' &= -H_\phi = 0.
\end{aligned}$$

(e) Solve the Hamilton-Jacobi equation.

Answer: The Hamilton-Jacobi equation is $S_t + H(\theta, \phi, S_\theta, S_\phi) = 0$. In this case that becomes

$$S_t + \frac{S_\theta^2}{2mR^2} + \frac{S_\phi^2}{2mR^2\sin^2\theta} - mgR\cos\theta = 0$$

Assuming an initial separation of variables $S(t,\theta,\phi) = W(\theta,\phi) - Et$. the Hamilton-Jacobi equation reduces to

$$\frac{W_\theta^2}{2mR^2} + \frac{W_\phi^2}{2mR^2\sin^2\theta} - mgR\cos\theta = E,$$

where E is the total energy. Next we try to find a solution of the form $W(\theta,\phi) = f(\theta) + g(\phi)$. We need

$$\frac{f'(\theta)^2}{2mR^2} + \frac{g'(\phi)^2}{2mR^2\sin^2\theta} - mgR\cos\theta = E.$$

Considering that θ and ϕ are independent variables, we must have $g'(\phi) = \alpha$, where α is a constant. then $g'(\phi) = \alpha\phi$, and

$$\frac{f'(\theta)^2}{2mR^2} + \frac{\alpha^2}{2mR^2\sin^2\theta} - mgR\cos\theta = E.$$

we solve for $f'(\theta)$, getting

$$f'(\theta) = \sqrt{2mR^2\left[E + mgR\cos\theta\right] - \frac{\alpha^2}{\sin^2\theta}}$$

or

$$f(\theta) = \int \sqrt{2mR^2\left[E + mgR\cos\theta\right] - \frac{\alpha^2}{\sin^2\theta}}\, d\theta.$$

Hence

$$S(t,\theta,\phi) = \int \sqrt{2mR^2\left[E + mgR\cos\theta\right] - \frac{\alpha^2}{\sin^2\theta}}\, d\theta + \alpha\phi - Et.$$

(f) Solve for $\theta(t)$ and $\phi(t)$ in terms of unevaluated integrals.

Answer: We solve by introducing new constants $\tau = S_E$ and $\beta = S_\alpha$. In the first case we get

$$\tau + t = \int \frac{mR^2\, d\theta}{\sqrt{2mR^2\left[E + mgR\cos\theta\right] - \frac{\alpha^2}{\sin^2\theta}}}$$

Assuming this integral can be evaluated, we can find θ as a function of t. In the second case, we get

$$\phi - \gamma = \int \frac{\alpha d\theta}{\sin^2\theta\sqrt{2mR^2\left[E + mgR\cos\theta\right] - \frac{\alpha^2}{\sin^2\theta}}}$$

This enables us to solve for ϕ as a function of θ.

4. Consider the functional $\mathcal{F}(u) = \int_0^b u'(x)^2\, dx$.
   (a) Find all extremals for $\mathcal{F}(u)$ subject to the constraint

$$\int_0^b u(x)^2\, dx = 1, \tag{1}$$

and the boundary conditions $u(0) = u(b) = 0$.

Answer: For any constrained extremal u there is a real number λ such that u is an extremal of

$$\mathcal{G}(u) = \int_0^b \left[u'(x)^2 + \lambda u(x)^2\right]\, dx.$$

u must solve the Euler Lagrange equation, which reduces to

$$u'' - \lambda u = 0.$$

There are three cases:

- $\lambda = -\alpha^2 < 0$. The general solution is $u(x) = Ae^{\alpha x} + Be^{-\alpha x}$. The only solution which solves the boundary conditions is $u(x) = 0$, which cannot be a contrained extremum.
- $\lambda = 0$. The general solution is $u(x) = Ax + B$. Again, the only solution which satisfies the boundary conditions is $u = 0$.
- $\lambda = \alpha^2 > 0$. The general solution is $u(x) = A\sin\alpha x + B\cos\alpha x$. The first boundary condition gives $0 = u(0) = B$. Hence $u(x) = A\sin\alpha x$. The second boundary condition requires $0 = u(b) = A\sin\alpha b$. Since we cannot have $A = 0$, we must have $\sin\alpha b = 0$. This means that $\alpha b = k\pi$ or $\alpha = k\pi/b$ for some integer k. To satisfy the constraint, we must have

$$1 = A^2 \int_0^b \sin^2\left(\frac{k\pi x}{b}\right)\, dx = A^2\frac{b}{2}.$$

Consequently, our extremals are

$$u_k(x) = \sqrt{\frac{2}{b}}\sin\left(\frac{k\pi x}{b}\right) \quad \text{for } k \text{ any nonzero integer.}$$

(b) Among all of the extremals you found in part (a), which could be a minimizer for $\mathcal{F}(u)$ subject to the constraints in part (a)? A proof is not necessary.

Answer: We have

$$u_k'(x) = -\left(\frac{k\pi}{b}\right) u_k(x),$$

so, using the constraint in (1),

(2) $$\int_0^b u_k'(x)^2\, dx = \left(\frac{k\pi}{b}\right)^2 \int_0^b u_k(x)^2\, dx = \left(\frac{k\pi}{b}\right)^2.$$

This can be a minimum only for $k = \pm 1$.

(c) Assuming that your answer in part (b) is really the minimizer of the constrained problem in part (a), prove the following version of Poincaré's inequality:

$$\int_0^b v(x)^2\, dx \le \frac{b^2}{\pi^2}\int_0^b v'(x)^2\, dx \quad \text{for all } v \in C_0^1(0,b).$$

(*Hint:* If $v \in C_0^1(0,b)$ is not the zero function, can you find a constant c such that $u = cv$ satisfies the constraint in part (a)?)

Answer: Suppose that $v \in C_0^1(0,b)$ is not the zero function, and set

$$c = 1/\left(\int_0^b v(x)^2\,dx\right)^{1/2}.$$

Then $u = cv$ satisfies the constraint in (1). Therefore, assuming that u_1 as found in part(a) is a global minimum of the constrained problem, we must have

$$\int_0^b u'(x)^2\,dx \geq \int_0^b u_1'(x)^2\,dx. \tag{3}$$

Hence

$$\begin{aligned}
\int_0^b v(x)^2\,dx &= \frac{1}{c^2}\int_0^b u(x)^2\,dx && (v = \tfrac{1}{c}u)\\
&= \frac{1}{c^2}\int_0^b u_1(x)^2\,dx && (u \text{ and } u_1 \text{ satisfy (1)})\\
&= \frac{1}{c^2}\frac{b^2}{\pi^2}\int_0^b u_1'(x)^2\,dx && (\text{by (2)})\\
&\leq \frac{1}{c^2}\frac{b^2}{\pi^2}\int_0^b u'(x)^2\,dx && (\text{by (3)})\\
&= \frac{b^2}{\pi^2}\int_0^b v'(x)^2\,dx && (u = cv)
\end{aligned}$$