1 Homework #1

1. Find a function $g \in C^\infty(\mathbf{R})$ which satisfies

$$g(x) = \begin{cases} 0 & x \le -1 \\ 1 & x \ge 0 \end{cases} \tag{1}$$

Hint: Suppose $\phi \in C^\infty(\mathbf{R})$ has supp $\phi \subset [-1, 0]$ and $\phi(x) \ge 0$ for all x. Consider the properties of $\psi(x) = \int_{-\infty}^{x} \phi(y)dy$.

Recall the example of a $C^\infty(\mathbf{R})$ given in class:

$$h(x) = \begin{cases} 0 & x \le 0 \\ e^{-\frac{1}{x}} & x > 0 \end{cases} \tag{2}$$

Following the hint, we will modify $h(x)$ to meet the requirements of ϕ. Consider the polynomial $p(x) = -x^2 - x = -x(x + 1)$. Observe that $p(0) = p(-1) = 0$, and $p(x) > 0$ for $x \in (-1, 0)$. Let

$$\phi(x) = h(p(x))$$

Since ϕ is the composition of two C^∞ functions, it will also be C^∞. For $x \ge 0$ or $x \le -1$, $p(x) \le 0$, so $h(p(x)) = 0$. Therefore, supp $\phi \subset [0, 1]$.

Now, let

$$\psi(x) = \int_{-\infty}^{x} \phi(y)dy.$$

By FTC, since $\phi \in C^\infty$, so will ψ.
For $x \le -1$, $\psi(x) = 0$ since $\phi(y) = 0$ for all $y \le -1$.
For $x \in (-1, 0)$, $\psi(x) = \int_{-1}^{x} \phi(y)dy > 0$, since $\phi(y) > 0$ for $y \in (-1, 0)$.
For $x \ge 0$, $\psi(x) = \int_{-1}^{0} \phi(y)dy$. To scale this to be equal to one, let $C = \int_{-1}^{0} \phi(y)dy$ and let $g(x) = \frac{\psi(x)}{C}$.

2. Let $I = [a, b]$ where $a < b$. Suppose $\epsilon > 0$. Find a function $\phi \in C^\infty(\mathbf{R})$ such that $\phi(x) = 1$ if $x \in$ I and supp $\phi \subset [a - \epsilon, b + \epsilon]$.

Using the $g(x)$ we created from the previous problem, define ϕ by:

$$\phi(x) = \begin{cases} g(\frac{x-a}{\epsilon}) & x \le \frac{a+b}{2} \\ g(\frac{b-x}{\epsilon}) & x \ge \frac{a+b}{2} \end{cases} \tag{3}$$

Then $\phi(a - \epsilon) = g(-1) = 0, \phi(a) = g(0) = 1$, and $\phi(x) = 1$ for $a < x \le (a + b)/2$. Similarly, $\phi(b) = g(0) = 1, \phi(b + \epsilon) = g(-1) = 0$, and $\phi(x) = 1$

for $(a+b)/2 \geq x < b$. Also $\phi(x)$ will be equal to zero if $x \leq a-\epsilon$ or $x \geq b+\epsilon$.

Therefore, ϕ satisfies the desired characteristics and has supp $\subset [a-\epsilon, b+\epsilon]$.

3. Analyze the variational problems corresponding to the following functionals, where in each case $y(0) = 0, y(1) = 1$. Does a min/max exist?

   (a) $\int_0^1 y' dx$
   $\int_0^1 y' dx = y(1) - y(0) = 1$. Therefore the functional is constant and there is no max or min.

   (b) $\int_0^1 yy' dx$
   Integration by parts gives us $\int_0^1 yy' dx = y^2|_0^1 - \int_0^1 yy' dx$, which means

   $$\int_0^1 yy' dx = \frac{1}{2}(y^2(1) - y^2(0)) = \frac{1}{2}$$

   Again the functional is constant, so there is no max or min.

   (c) $\int_0^1 xyy' dx$
   Again, using integration by parts, we find:
   $\int_0^1 xyy' dx = \frac{1}{2}(xy^2|_0^1 - \int_0^1 y^2 dx) = \frac{1}{2}(1 - \int_0^1 y^2 dx)$.

   As $y^2(x) \geq 0$, $\int_0^1 y^2 dx \geq 0$. So the maximal value our functional can achieve is $1/2$, but we do not know if it can achieve it.

   Consider the function $y(x) = x^k$. Then

   $$\int_0^1 xyy' dx = \frac{1}{2}\left(1 - \int_0^1 x^{2k} dx\right) = \frac{1}{2}\left(1 - \frac{1}{2k+1}\right)$$

   As k gets larger, this value gets closer to $1/2$. To actually equal $1/2$, $k \to \infty$, but in terms of our function, this would mean $y(x) = 0$ for $x \in [0, 1)$ and $y(1) = 1$. However, this function is not continuous, and thus the functional does not have a maximum when restricted to continuous functions with designated boundary conditions.

   As far as a minimum goes, $\int_0^1 y^2 dx$ can be as large as we like, making our functional value as small as we'd like, meaning a minimum also does not exist.

   For a specific example, consider $y(x) = kx(x-1)$. Then $\int_0^1 y^2 dx = k^2/30$.

4. Find the extremals of the following functionals:

(a) $\int_a^b (y^2 + y'^2 - 2y\sin(x))dx$
$F(x, y, y') = y^2 + y'^2 - 2y\sin(x)$, so the Euler equation is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = (2y - 2\sin(x)) - (2y'') = 0.$$

$$\implies y''(x) = y(x) - \sin(x)$$

Solving this differential equation gives us the general solution

$$y(x) = C_1 e^x + C_2 e^{-x} + \frac{\sin(x)}{2}$$

for constants C_1, C_2.

(b) $\int_a^b \frac{y'^2}{x^3} dx$
The associated Euler equation is

$$\frac{d}{dx}\left(\frac{2y'}{x^3}\right) = 0$$

$$\implies \frac{2y'}{x^3} = C_1$$
$$\implies 2y' = C_1 x^3$$
$$\implies y = C_2 x^4 + C_3$$

(c) $\int_a^b (y^2 - y'^2 - 2y\cosh(x))dx$ The associated Euler equation is

$$(2y - 2\cosh(x)) - (-2y'') = 0$$

$$\implies y'' = -y + \cosh(x)$$

The general solution to this differential equation is

$$y(x) = \cosh(x)/2 + C_1\cos(x) + C_2\sin(x).$$