Homework #3 Due February 7.

- Read Sections 5 and 9 in Gelfand & Fomin.

1. Suppose that $F(x,u,p) = u^2p^2$, and $\mathcal{F}(u) = \int_a^b F(x,u(x),u'(x))\,dx$.

(a) Show that all extremals are parabolas.

(b) Find u_0, the extremal satisfying the boundary conditions $u_0(a) = 1$ and $u_0(b) = 0$, where $a < b$. Sketch a graph of the extremal.

(c) Compute $\mathcal{F}(u_0)$ and $\mathcal{F}(u)$ where $u(x)$ is the linear function satisfying $u(a) = 1$ and $u(b) = 0$.

2. The area of the graph of a function $u(x,y)$ defined on a set Ω is given by the functional

$$\mathcal{A}(u) = \int\int_\Omega \sqrt{1 + u_x^2 + u_y^2}\,dxdy.$$

Find all extremals of the form $u(x,y) = f(x) + g(y)$, where f and g are functions of one variable.

3. Consider the functional

$$\mathcal{F}(u) = \int_a^b F(u(x),u'(x))\,dx. \qquad (*)$$

Since $F_x = 0$, we know that the quantity $G = pF_p - F$ is constant when evaluated on an extremal. The source of the expression G has been the subject of some mystery. We did relate it to the law of conservation of energy, but that may not have satisfied you. This exercise will explore another way in which the expression G might have been discovered.

(a) Interchange the roles of the independent and dependent variables in the integral in $(*)$ by making the substitution $y = u(x)$. If you let $v = u^{-1}$, then $x = v(y)$. Show that the functional in $(*)$ can be written as

$$\mathcal{F}(u) = \mathcal{H}(v) = \int_{a'}^{b'} H(y,v(y),v'(y))\,dy,$$

where $u(a) = a'$, $u(b) = b'$, and $H(y,v,q) = qF(y,1/q)$.

(b) You will notice that $H_v = 0$. Use this to show that G is constant.

4. A surface S in parametric form is defined by the three equations

$$x = f(u,v), \quad y = g(u,v), \quad \text{and} \quad z = h(u,v),$$

where $(u,v) \in U$ and U is an open subset of $\mathbf{R^2}$.

(a) Show that the element of arc length on S is given by the expression

$$ds^2 = E(u,v)\,du^2 + 2F(u,v)\,dudv + G(u,v)\,dv^2,$$

where $E = f_u^2 + g_u^2 + h_u^2$, $F = f_u f_v + g_u g_v + h_u h_v$, and $G = f_v^2 + g_v^2 + h_v^2$.

(b) A sphere of radius R can be paramerterized using the longitude θ and the polar angle ϕ (the compliment of the latitude) as parameters. The equations are

$$x = R\cos\theta\sin\phi, \quad y = R\sin\theta\sin\phi, \quad \text{and} \quad z = R\cos\phi. \qquad (**)$$

Show that $ds^2 = R^2[d\phi^2 + \sin^2\phi\,d\theta^2]$.

(c) If we limit ourselves to curves which can be parameterized by the polar angle ϕ, with $\theta = w(\phi)$, $a \le \phi \le b$, show that the length of the curve γ is given by

$$\mathcal{F}(w) = \int_\gamma ds = \int_a^b F(\phi, w, w')\,d\phi,$$

where $F(\phi, w, p) = R\sqrt{1 + p^2\sin^2\phi}$.

(d) Integrate the Euler equation for this functional and use the parametric equations for the sphere (**) to show that there are constants A, B, and C such that the extremal curve lies in the plane through the origin defined by $Ax + By + Cz = 0$. Hence you will have proved that the geodesics on the sphere are great circles. The needed integral is hard to evaluate. Use a computer algebra system or a table of integrals, or come to see me.