

1 Homework #3

1. Suppose $F(x, u, p) = u^2 p^2$, and $\mathcal{F}(u) = \int_a^b F(x, u(x), u'(x)) dx$

(a) Show all extremals are parabolas.

The Euler equation for this functional is

$$2up^2 - \frac{d}{dx}(2u^2p) = 0$$

$$2up^2 - (4up^2 + 2u^2p') = 0$$

$$u(u')^2 + u^2u'' = 0$$

$$u((u')^2 + uu'') = 0$$

$$\implies u = 0 \text{ or } (u')^2 + uu'' = 0$$

Notice that $(u^2)' = 2uu'$ and $(u^2)'' = 2(u')^2 + 2uu''$. Therefore, either $u = 0$ or

$$(u^2)'' = 0 \implies u(x)^2 = C_1x + C_2$$

(b) With $u_0(a) = 1, u_0(b) = 0$

$$1^2 = C_1a + C_2, 0^2 = C_1b + C_2$$

$$\implies C_1 = \frac{1}{a-b}, C_2 = \frac{-b}{a-b}$$

$$u_0(x)^2 = \frac{x-b}{a-b}$$

(c)

$$u_0(x)^2 = \frac{x-b}{a-b}$$

$$2u_0u_0' = \frac{1}{a-b} \implies (u_0')^2 = \frac{1}{(2u_0(a-b))^2}$$

Thus

$$\mathcal{F}(u_0) = \int_a^b u_0^2 (u_0')^2 dx \quad (1)$$

$$= \int_a^b \frac{u_0^2}{4u_0^2(a-b)^2} dx \quad (2)$$

$$= \int_a^b \frac{1}{4(a-b)^2} dx \quad (3)$$

$$= \frac{1}{4(a-b)^2} x \Big|_a^b \quad (4)$$

$$= -\frac{1}{4(a-b)} \quad (5)$$

For the linear function, $u = \frac{x-b}{a-b}$

$$\mathcal{F}(u) = \int_a^b u^2 (u')^2 dx \quad (6)$$

$$= \int_a^b \frac{(x-b)^2}{(a-b)^2} \frac{1}{(a-b)^2} dx \quad (7)$$

$$= \frac{1}{(a-b)^4} \int_a^b (x-b)^2 dx \quad (8)$$

$$= \frac{1}{(a-b)^4} \frac{(x-b)^3}{3} \Big|_a^b \quad (9)$$

$$= -\frac{1}{3(a-b)} \quad (10)$$

2.

$$\mathcal{A}(u) = \int \int_{\Omega} \sqrt{1 + u_x^2 + u_y^2} dx dy$$

Find all extremals of the form $u(x, y) = f(x) + g(y)$.

The Euler equation for this functional is

$$A_u - \frac{\partial}{\partial x}(A_{u_x}) - \frac{\partial}{\partial y}(A_{u_y}) = 0$$

$$A_u = 0,$$

$$A_{u_x} = \frac{f'(x)}{\sqrt{1 + f'(x)^2 + g'(y)^2}}, A_{u_y} = \frac{g'(y)}{\sqrt{1 + f'(x)^2 + g'(y)^2}}$$

$$\frac{\partial}{\partial x} A_{u_x} = \frac{f''(x) [1 + f'(x)^2 + g'(y)^2] - f'(x)^2 f''(x)}{(1 + f'(x)^2 + g'(y)^2)^{3/2}}$$

$$\frac{\partial}{\partial y} A_{u_y} = \frac{g''(y) [1 + f'(x)^2 + g'(y)^2] - g'(y)^2 g''(y)}{(1 + f'(x)^2 + g'(y)^2)^{3/2}}$$

Plugging these expressions into the Euler equation, we get

$$f''(x) [1 + f'(x)^2 + g'(y)^2] - f'(x)^2 f''(x) + g''(y) [1 + f'(x)^2 + g'(y)^2] - g'(y)^2 g''(y) = 0$$

$$f''(1 + g'^2) + g''(1 + f'^2) = 0 \implies \frac{f''(x)}{1 + f'(x)^2} = -\frac{g''(y)}{1 + g'(y)^2}$$

Because the left side of this equation only depends on x and the right only on y , they must both be equal to some constant, C . Now, we can solve for f and g .

$$\frac{f''(x)}{1 + f'(x)^2} = C$$

If $C = 0$, $f''(x) = 0 \implies f(x) = C_1 x + C_2$. If $C \neq 0$, let $u = f'$,

$$u' = C(1 + u^2)$$

This is a separable equation solved by

$$\begin{aligned} \int \frac{du}{1 + u^2} &= \int C dx \\ \arctan(u) &= Cx + C_1 \\ u &= \tan(Cx + C_1) \end{aligned}$$

Now, since $u = f'$,

$$f' = \tan(Cx + C_1)$$

$$f = \frac{1}{C} \ln(\cos(Cx + C_1)) + C_2$$

Similarly,

$$g(y) = C_3 y + C_4$$

or

$$g = -\frac{1}{C} \ln(\cos(-Cy + C_3)) + C_4$$

3.

$$\mathcal{F}(u) = \int_a^b F(u(x), u'(x)) dx$$

(a) $y = u(x)$, let $v = u^{-1}$, $x = v(y)$. By the inverse function theorem,

$$v'(y) = (u^{-1})'(y) = \frac{1}{(u'(x))}$$

$dx = v'(y)dy$. Plugging this into the functional,

$$\mathcal{F}(u) = \int_a^b F(u(x), u'(x)) dx \quad (11)$$

$$= \int_{u(a)}^{u(b)} F(y, \frac{1}{v'(y)}) v' dy \quad (12)$$

$$= \int_{a'}^{b'} H(y, v, v') dy \quad (13)$$

where $u(a) = a'$, $u(b) = b'$ and $H(y, v, q) = qF(y, 1/q)$.

(b) $H_v = 0$. Therefore the Euler equation for this functional is

$$-\frac{d}{dy} H_q = 0$$

Thus, $H_q = C$, for some constant.

$$H_q = F(y, 1/q) + qF_q(y, 1/q)(-1/q^2) \quad (14)$$

$$= F(u, p) - pF_p(y, 1/q) \quad (15)$$

Thus, $pF_p - F = C$.

4. $x = f(u, v)$, $y = g(u, v)$, $z = h(u, v)$

(a) $ds^2 = dx^2 + dy^2 + dz^2$.

$$dx = f_u du + f_v dv \implies dx^2 = f_u^2 du^2 + 2f_u f_v dudv + f_v^2 dv^2 \quad (16)$$

$$dy = g_u du + g_v dv \implies dy^2 = g_u^2 du^2 + 2g_u g_v dudv + g_v^2 dv^2 \quad (17)$$

$$dz = h_u du + h_v dv \implies dz^2 = h_u^2 du^2 + 2h_u h_v dudv + h_v^2 dv^2 \quad (18)$$

$$(19)$$

Thus,

$$\begin{aligned}
ds^2 &= (f_u^2 + g_u^2 + h_u^2)du^2 + 2(f_u f_v + g_u g_v + h_u h_v)dudv + (f_v^2 + g_v^2 + h_v^2)dv^2 \\
&= Edu^2 + 2Fdudv + Fdv^2
\end{aligned}$$

(b)

$$f_\theta = -R \sin \theta \sin \phi, f_\phi = R \cos \theta \cos \phi \quad (20)$$

$$g_\theta = R \cos \theta \sin \phi, g_\phi = R \sin \theta \cos \phi \quad (21)$$

$$h_\theta = 0, f_\phi = -R \sin \phi \quad (22)$$

$$E = R^2 \sin^2 \phi \quad (23)$$

$$F = 0 \quad (24)$$

$$G = R^2 \quad (25)$$

$$(26)$$

Therefore, $ds^2 = R^2(d\phi^2 + \sin^2 \phi d\theta^2)$.

(c)

$$ds^2 = R^2 \sin^2 \phi (w')^2 d\phi^2 + R^2 d\phi^2 = R^2(1 + (w')^2 \sin^2 \phi) d\phi^2$$

$$ds = R \sqrt{1 + (w')^2 \sin^2 \phi} d\phi$$

$$\mathcal{F}(u) = \int_\gamma ds = \int_a^b R \sqrt{1 + (w')^2 \sin^2 \phi} d\phi$$

(d) Presented in class