Homework #5 Due February 21.

- Read Section 12 in Gelfand & Fomin.

1. Consider the cylinder S in $\mathbf{R}^3$ defined by the equation $x^2 + y^2 = a^2$. The points $A = (a, 0, 0)$ and $B = (a\cos\alpha, a\sin\alpha, b)$ both lie on S. Find the geodesics joining them.

2. Let S be a surface of revolution about the $z-$axis. It is defined by an equation of the form $x^2 + y^2 = f(z)$. Let γ be a geodesic on S and let $u(t) = (x(t), y(t), z(t)$ be its parameterization by arc length.

(a) Show that $xy' - yx' = c_1$, where c_2 is a constant.

(b) Cylindrical coordinates r, θ, and z are related to the usual cartesian coordinates x, y, and z by the equations $x = r\cos\theta$ and $y = r\sin\theta$. Show that the element of arc length in $\mathbf{R}^3$ is given by $ds^2 = dr^2 + r^2 d\theta^2 + dz^2$.

(c) Express the parameterization for the geodesic γ in cylindrical coordinates $u(t) = (r(t), \theta(t), z(t))$, and show that $r^2\theta' = c_2$. Do this using the Euler equations for u. Then show that $c_2 = c_1$ where c_1 is the constant in part a).

3. Consider a surface S in $\mathbf{R}^3$ defined by

$$g(x, y, z) = 0. \tag{1}$$

A thread has its end fixed at the points P_1 and P_2 on S, and its position is given by the parameterization

$$u(t) = (x(t), y(t), z(t)) \quad 0 \le t \le L.$$

Assume that this is the parameterization by arc length, so

$$|u'(t)| = 1. \tag{2}$$

The thread has assumed an equilbrium position under the force of gravity (which points in the negative z direction). This means that the potential energy is minimized, which is equivalent to

$$\int_0^L z(t)\,dt \tag{3}$$

being minimized. Describe how you would proceed to find u. In particular show the differential equations that u must satisfy. It is not necessary to solve the equations.