1 Homework #5

1. We can parameterize the cylinder by

$$\mathbf{r}(\theta, z) = (a\cos\theta, a\sin\theta, z)$$

Then for a curve on the surface, we define the arclength functional

$$L(\theta, z) = \int_a^b \sqrt{E\theta'^2 + 2F\theta' z' + Gz'^2}dt$$

where E, F, G are coefficients of the first fundamental form of the surface, i.e.

$$E = \mathbf{r}_\theta \cdot \mathbf{r}_\theta = a^2$$
$$F = \mathbf{r}_\theta \cdot \mathbf{r}_z = 0$$
$$G = \mathbf{r}_z \cdot \mathbf{r}_z = 1$$

Thus, the Euler equations for our functional are:

$$\frac{d}{dt}\frac{a^2\theta'}{\sqrt{a^2\theta'^2 + z'^2}} = 0$$
$$\frac{d}{dt}\frac{z'}{\sqrt{a^2\theta'^2 + z'^2}} = 0$$

$$\Longrightarrow \frac{a^2\theta'}{\sqrt{a^2\theta'^2 + z'^2}} = C_1$$
$$\frac{z'}{\sqrt{a^2\theta'^2 + z'^2}} = C_2$$

Combining these two expressions,

$$z = C_3\theta + C_4$$

However, we need to use boundary conditions, that the geodesic goes through $A = (a, 0, 0), B = (a\cos\alpha, a\sin\alpha, b)$. When $\theta = 0$, $z = C_4 = 0$. Thus, $z = C_3\theta$.

Also, when $\theta = \alpha, z = C_3\alpha = b \Longrightarrow C_3 = b/\alpha$.

$$z = \frac{b}{\alpha}\theta$$

2. (a) Consider the following functional

$$\mathcal{F} = \int |u'(t)| + \lambda(t)(x^2 + y^2 - f(z))dt$$

For γ to be a geodesic, u must satisfy the Euler equations for this functional, i.e.

$$2\lambda x - \frac{d}{dt}\frac{x'}{|u'|} = 0$$

$$2\lambda y - \frac{d}{dt}\frac{y'}{|u'|} = 0$$

$$-\lambda\frac{df}{dz} - \frac{d}{dt}\frac{x'}{|u'|} = 0$$

Using the fact that u is parameterized by arc length, we get

$$2\lambda x = x''$$

$$2\lambda y = y''$$

$$-\lambda\frac{df}{dz} = z''$$

Multiplying first equation by y, second by x, we find $yx'' = xy''$. But notice

$$\frac{d}{dt}(xy' - yx') = xy'' + x'y' - y'x' - yx'' = xy'' - yx''$$

Therefore, $\frac{d}{dt}(xy' - yx') = 0$, i.e. $xy' - yx' = C_1$.

(b)

$$dx = \cos\theta dr - r\sin\theta d\theta$$

$$dy = \sin\theta dr + r\cos\theta d\theta$$

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$= dr^2 + r^2 d\theta^2 + dz^2$$

(c) Let

$$\mathcal{G} = \int_a^b \sqrt{dr^2 + r^2 d\theta^2 + dz^2} + \lambda(t)(r^2 - f(z))dt$$

The θ-Euler equation for this functional is

$$\frac{r^2\theta'}{\sqrt{dr^2 + r^2 d\theta^2 + dz^2}} = C_2$$

Since u is parameterized by arclength, $r^2\theta' = C_2$.

Now, $r^2 = x^2 + y^2 = f(z)$, and $\tan\theta = y/x$. Thus

$$\sec^2\theta\theta' = \frac{xy' - yx'}{x^2} \Longrightarrow \theta' = \frac{xy' - yx'}{x^2 + y^2}$$

Thus, $c_2 = r^2\theta' = xy' - yx' = C_1$.

3. This is an extremal problem with two constraints, $g = 0$ and $\int_0^L \sqrt{x'^2 + y'^2 + z'^2}dt = L$. Using Lagrange multipliers, we will find extremum for the new functional:

$$\mathcal{F} = \int_0^L z(t) + \lambda_1 g + \lambda_2\sqrt{x'^2 + y'^2 + z'^2}$$

The Euler equations for this functional are:

$$\lambda_1 g_x - \frac{d}{dt}\frac{\lambda_2 x'}{\sqrt{x'^2 + y'^2 + z'^2}} = 0$$
$$\lambda_1 g_y - \frac{d}{dt}\frac{\lambda_2 y'}{\sqrt{x'^2 + y'^2 + z'^2}} = 0$$
$$z' + \lambda_1 g_z - \frac{d}{dt}\frac{\lambda_2 z'}{\sqrt{x'^2 + y'^2 + z'^2}} = 0$$

Bust $\sqrt{x'^2 + y'^2 + z'^2} = 1$, so this reduces to

$$\lambda_1 g_x = \frac{d}{dt}(\lambda_2 x')$$
$$\lambda_1 g_y = \frac{d}{dt}(\lambda_2 y')$$
$$z' + \lambda_1 g_z = \frac{d}{dt}(\lambda_2 z')$$