1 Homework #7

1. Let xy be coordinates in the vertical plane, and let the origin be the center of the starting circle. Thus, the starting point, (a, a'), must satisfy the equation for the circle, $a^2 + a'^2 = r_1^2$. The equation for the second circle will be given by $(x-d)^2 + (y-d')^2 = r_2^2$.

   Conservation of energy:

   $$1/2mv^2 = mg(a' - y)$$

   $$v = \sqrt{2g(a'-y)}$$

   The transition time

   $$T[y] = \int_a^b \frac{\sqrt{1+y'^2}}{v}dx = \int_a^b \frac{\sqrt{1+y'^2}}{\sqrt{2g(a'-y}}dx$$

   subject to the boundary conditions, $y(a) = a', y(b) = b'$ and transerversality conditions.

   Let $\phi(x) = \pm\sqrt{r_1^2 - x^2}$, $\psi(x) = \pm\sqrt{r_2^2 - (x-d)^2}$. Then our transversality conditions, as on p. 61, will reduce to:
   $y'(a) = -1/\phi'(a) = \pm a'/a, y'(b) = -1/\psi'(b) = \pm\frac{b'-d'}{b-d}$.
   The Euler equation is:

   $$-1/2\frac{\sqrt{1+y^2}}{(2g(a'-y))^{3/2}}(-2g) - \frac{d}{dx}\frac{y'}{\sqrt{2g(a'-y)}\sqrt{1+y'^2}} = 0$$

   $$\implies y'^2 - 2a'y'' + 2yy'' + 1 = 0$$

   which has the Brachistochrone solution

   $$x(t) = k(t - \sin t) + a, y(t) = a' - k(1 - \cos t)$$

   with the above boundary and transversality conditions.

2. $$J[y,z] = \int_a^b \sqrt{1+y'^2+z'^2}dx$$

   The Euler equations for this functional are:

   $$\frac{y'}{\sqrt{1+y'^2+z'^2}} = C_1$$

$$\frac{z'}{\sqrt{1+y'^2+z'^2}} = C_2$$

Dividing these two expressions, we get that both $y', z' = C$, $y = C_3x + C_4, z = C_5x + C_6$.

Then our extremals are straight lines. The boundary conditions satisfy $a_1 = C_3a + C_4, a_2 = c_5a + C_6$. For the transversality conditions, it makes sense that the curve minimizing the distance between two surfaces should intersect them orthogonally. In this case, that would mean $(1, y'(a), z'(a))$ should be in the same direction as the normal to the first surface, $(\phi_x(a, a_1), \phi_y(a, a_1), -1)$ and similalry for the second .

For the actual trnasversality conditions, the surfaces in this problem are given by $\phi(x, y) - z = 0, \psi(x, y) - z = 0$ rather than, on p. 61, as $\phi(y, z) = x, \psi(y, z) = x$. The transversality conditions should change from those given on p61 in the following way:

$$F_{y'} - \frac{\phi_y}{\phi_x}(F - y'F_{y'} - z'F_{z'})|_{x=a} = 0$$
$$F_{z'} + \frac{1}{\phi_x}(F - y'F_{y'} - z'F_{z'})|_{x=a} = 0$$
$$\vdots$$

After simplification, this reduces to $y'(a) = \phi_y(a, a_1)/\phi_x(a, a_1), z'(a) = -1/\phi_x(a, a_1)$ and similarly for the second point. Thus, the extremals are straight lines perpendicular to both surfaces.

3.

$$J[y, z] = \int_a^b f(x, y, z)\sqrt{1+y'^2+z'^2}dx$$

Here, again since the surfaces are defined by $\phi(x, y) = z, \psi(x, y) = z$, the transversality conditions will be:

$$F_{y'} - \frac{\phi_y}{\phi_x}(F - y'F_{y'} - z'F_{z'})|_{x=a} = 0$$
$$F_{z'} + \frac{1}{\phi_x}(F - y'F_{y'} - z'F_{z'})|_{x=a} = 0$$
$$\vdots$$

$$Fy' - \phi_y/\phi_x F(1+y'^2+z'^2-y'^2-z'^2) = F(y' - \phi_y/\phi_x) = 0 \Longrightarrow y'(a) = \phi_y(a, a_1)/\phi_x(a, a_1)$$

Similarly, $z'(a) = -1/\phi_x(a, a_1), y'(b) = \phi_y(b, b_1)/\phi_x(b, b_1), z'(b) = -1/\phi_x(b, b_1)$.

The vectors $(\phi_x, \phi_y, -1) = 1/\phi_x(1\phi_y/\phi_x, -1/\phi_x)$ is the normal vector to the surface $\phi(x, y) = z$, and this is the same as our transversality condition. Thus, at $x = a$ and b, extremals meet the surface perpendicularly.

4. The Euler equations for this functional are:

$$2z = 2y'', 2y = 2z''$$

i.e. $y = z'' = y''''$. Solving this equation, we get

$$y = C_1 e^x + C_2 e^{-x} + C_3 \sin x + C_4 \cos x$$
$$z = C_1 e^x + C_2 e^{-x} - C_3 \sin x - C_4 \cos x$$

Using $y(0) = z(0) = 0$, we get $C_4 = 0, C_1 = -C_2$.

The natural boundary conditions are:

$$y'|_{x_1} = 0, z'|_{x_1} = 0$$

i.e. $C_1(e^{x_1} + e^{-x_1}) + C_3 \cos x_1 = 0, C_1(e^{x_1} + e^{-x_1}) - C_3 \cos x_1 = 0$. Combining these tow equations, we get $C_3 = 0, C_1 = 0$, hence $y = z = 0$.

5. Euler equation:

$$2x^{2/3}y' = C$$

$y' = cx^{-2/3}$, when $x \neq 0 \Longrightarrow y = C_1 x^{1/3} + C_2$ when $x \neq 0$. With boundary conditions, $y(-1) = -1, y(1) = 1$ we get $C_1 = 1, C_2 = 0$. Thus, when $x \neq 0$, $y = x^{1/3}$ satisfies the Euler equation.

$F_{y'} = 2x^{2/3}y' = 2/3,\ F - y'F_{y'} = -1/9x^{-2/3}$.

For $x = 0$, $\lim_{x\to 0\pm} F_{y'} = 2/3$, $\lim_{x\to 0\pm}(F - y'F_{y'}) = -\infty$. Thus the Weistrass-Erdmann corner condition is satisfied and $y = x^{1/3}$ is the extremal.

6. Euler equation:

$$3y'^2 = C \Longrightarrow y = C_1 x + C_2$$

Therefore an extremal must be a straightline in each of its components. If (x_0, y_0) is a corner

$$y = \frac{y_0}{x_0} x, x < x_0 \tag{1}$$

$$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0), x > x_0 \tag{2}$$

Weiestrass-Erdmann conditions require:

$$\lim_{x \to 0+} F_{y'} = \lim_{x \to 0-} F_{y'}$$

$$\lim_{x \to 0+} (F - y' F_{y'}) = \lim_{x \to 0-} (F - y' F_{y'})$$

which give us

$$3(\frac{y_0}{x_0})^2 = 3(\frac{y_1 - y_0}{x_1 - x_0})^2$$

$$(\frac{y_0}{x_0})^3 = (\frac{y_1 - y_0}{x_1 - x_0})^3$$

Therefore, either $\frac{y_0}{x_0} = \pm \frac{y_1 - y_0}{x_1 - x_0}$. If the sign is $+$, y is continuous. If $-$, then the second condition is not met. Thus, there are no broken extremals.

7. Euler equation:

$$2(y' - 1)(y' + 1)2y' = C \Longrightarrow y'^3 - y' = C$$

Solving this equation, we get $y = C_1 x + C_2$.

Suppose we have a corner at (x_0, y_0). To meet the b.c.,

$$y = \frac{y_0}{x_0} x, x < x_0 \tag{3}$$

$$y - y_0 = \frac{2 - y_0}{4 - x_0}(x - x_0), x > x_0 \tag{4}$$

$F_{y'} = 4(y'^2 - 1)y', F - y' F_{y'} = (y'^2 - 1)(-3y'^2 - 1)$. Let $p_+ = y'_+, p_- = y'_-$ (the derivative of y to left and right of the corner point). To meet the Weistrass-Erdmann conditions,

$$(p_+^2 - 1)p_+ = (p_-^2 - 1)p_- \Longrightarrow (p_+ - p_-)(p_+^2 + p_-^2 + p_- p_+ - 1) = 0$$

$$(p_+^2 - 1)(3p_+^2 + 1) = (p_-^2 - 1)(3p_-^2 + 1) \Longrightarrow (p_+^2 - p_-^2)(3p_+^2 + 3p_-^2 - 2) = 0$$

Because we want a corner at (x_0, y_0), we want $p_+ \neq p_-$. Thus, from the first condition, we must get that $p_+^2 + p_-^2 + p_- p_+ - 1 = 0$. From the second, we get either $p_+ = -p_-$ or $3p_+^2 + 3p_-^2 - 2 = 0, (p_+^2 + p_-^2 = 2/3)$.

When $p_+ = -p_-$, and using $p_+^2 + p_-^2 + p_- p_+ - 1 = 0$, we get $p_+ = 1/\sqrt{2}$. Thus, the extremals are

$$y = \pm \frac{1}{\sqrt{2}} x, x < 2 \pm \sqrt{2} \tag{5}$$

$$y = 2 \pm (2\sqrt{2} - 1/\sqrt{2}x), x > 2 \pm \sqrt{2} \tag{6}$$

If $p_+^2 + p_-^2 = 2/3 \implies p_- p_+ = 1/3$, and after some algebra, can show this corresponds to the case when $p_- = p_+$, i.e. y' is continuous. Thus the only extremals with one corner are the ones found above.