1 Homework #8

1. $f(x) = e^x$.

   (a) $f''(x) = e^x > 0$, hence f is convex.
   Set $\xi = f'(x) = e^x$ and solve for x: $\ln \xi = x$. Then $f^*(\xi) = x\xi - f(x) = \xi \ln \xi - f(\ln \xi) = \xi(\ln \xi - 1)$.

   (b) Consider the function $g(y) = e^y - y$. $f'(y) = e^y - 1$ and has only one root when $y = 0$. $f''(y) = e^y, f''(0) > 0$. Hence, $y = 0$ is global minimum. Therefore, $f(y) \geq f(0) = 1 \Longrightarrow e^y - y \geq 1$.

   Let $y = x - n$. Then $e^{x-n} - x + n \geq 1$. Since $e^n > 0$, multiply expression by e^n:

   $$e^x - xe^n + ne^n \geq e^n \Longrightarrow xe^n \leq e^x + (n-1)e^n$$

2. (a) One way to show the matrix is positive definite is to show all the principle minors are positive, in this case show f_{xx} and $f_{xx}f_{yy} - f_{xy}^2 > 0$.

   $$f_{xx} = e^{\frac{x^2+y^2}{2}}(x^2+1) > 0$$

   $$f_{xx}f_{yy} - f_{xy}^2 = e^{\frac{x^2+y^2}{2}}(x^2+1)e^{\frac{x^2+y^2}{2}}(y^2+1) - e^{\frac{x^2+y^2}{2}}(x^2)e^{\frac{x^2+y^2}{2}}y^2 = e^{x^2+y^2}(x^2+y^2+1) > 0$$

   Therefore, $Hf(x,y)$ is positive definite.

   (b)

   $$\phi(x,y) = \nabla f(x,y) = (e^{\frac{x^2+y^2}{2}}x, e^{\frac{x^2+y^2}{2}}y)$$

   Observe that the points $(e,e),(e,-e)$ lie on the boundary of this region, and the straight line connecting them is a vertical line. This vertical line intersects the hortizontal axis at e, but the point on boundary of the region on the horizontal axis is $\sqrt{e}$. Thus, the line segment is not contained in the region, so it is not convex.

   (c) $\alpha'(t) = e^t + te^t = (t+1)e^t > 0$ when $t > 0$. Thus, $\alpha(t)$ is increasing. $\alpha(0) = 0, \lim_{t\to\infty} = \infty$, thus α is not bounded above. Altogether, and using that α is continuous, $\alpha(t)$ maps $[0,\infty)$ onto itself.

   $\xi = f_x = e^{\frac{x^2+y^2}{2}}x, \eta = f_y = e^{\frac{x^2+y^2}{2}}y$.

   $$\xi^2 + \eta^2 = e^{x^2+y^2}(x^2+y^2) = \alpha(x^2+y^2)$$

   Therefore, $x^2 + y^2 = \beta(\xi^2 + \eta^2)$.

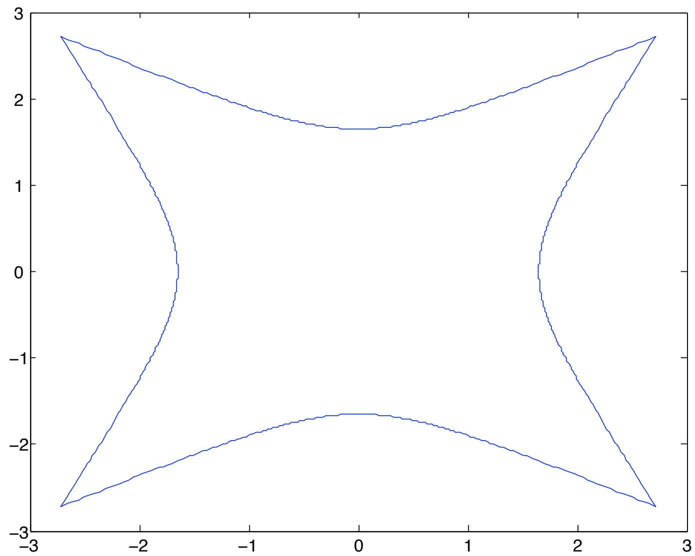

Figure 1: Region Ω^*

$$
\begin{aligned}
f^*(\xi,\eta) &= (x,y)\dot{(}\xi,\eta) - f(x,y) && (1)\\
&= (x,y)\dot{(}e^{\frac{x^2+y^2}{2}}x, e^{\frac{x^2+y^2}{2}}y) - e^{\frac{x^2+y^2}{2}} && (2)\\
&= (x^2+y^2)e^{\frac{x^2+y^2}{2}} - e^{\frac{x^2+y^2}{2}} && (3)\\
&= (\beta(\xi^2+\eta^2)-1)(e^{\frac{\beta(\xi^2+\eta^2)}{2}}) && (4)
\end{aligned}
$$

3. $\Pi = F_p = p + ku, p = \Pi - ku$

$$H(x,u,\Pi) = \Pi p - F = \Pi(\Pi - ku) - 1/2(\Pi - ku + ku) = 1/2\Pi^2 - \Pi ku$$

Euler equations:

$$\frac{du}{dx} = \frac{\partial H}{\partial \Pi} = \Pi - ku$$

$$\frac{d\Pi}{dx} = -\frac{\partial H}{\partial u} = \Pi k$$

$\Longrightarrow \Pi(x) = e^{kx}$, plugging this into the first equation for u, we get

$$u(x) = 1/(2k)e^{kx} + Ce^{-kx}$$

4. (a) $u(x)$ the potential energy, $\frac{d}{dx}u = F = -kx$. Thus, $u(x) = 1/2kx^2 + C$, take $C = 0$. Lagrangian, $L(x,p) = T - V = 1/2mp^2 - 1/2kx^2$.

(b)
$$\Pi = \frac{\partial L}{\partial p} = mp, p = \Pi/m$$

$$H(t,x,\Pi) = \Pi p - L = \Pi^2/m - 1/2m(\Pi/m)^2 + 1/2kx^2 = 1/2\Pi^2/m + 1/2kx^2$$

Π is momentum and H is total energy.

(c)
$$\frac{\partial H}{\partial \Pi} = \frac{dx}{dt} = \Pi/m$$
$$\frac{\partial H}{\partial x} = -\frac{d\Pi}{dt} = kx$$

$$\frac{d}{dt}(\frac{dx}{dt}) = \frac{d\Pi}{dt}\frac{1}{m} \Longrightarrow x'' = \frac{-kx}{m}$$

Therefore, since $k > 0$, the extremals are

$$x(t) = A\sin(\sqrt{\frac{k}{m}}t) + B\cos(\sqrt{\frac{k}{m}}t)$$