Homework #3

Due February 10.

Read Sections 9 and 12 in Gelfand & Fomin.

1. Suppose that $F(x,u,p) = u^2p^2$, and $\mathcal{F}(u) = \int_a^b F(x,u(x),u'(x))\,dx$.
   (a) Show that all extremals are parabolas.
   (b) Find u_0, the extremal satisfying the boundary conditions $u_0(a) = 1$ and $u_0(b) = 0$, where $a < b$. Sketch a graph of the extremal.
   (c) Compute $\mathcal{F}(u_0)$ and $\mathcal{F}(u)$ where $u(x)$ is the linear function satisfying $u(a) = 1$ and $u(b) = 0$.
2. Consider the functional

$$\mathcal{F}(u) = \int_a^b F(u(x),u'(x))\,dx. \qquad (*)$$

   Since $F_x = 0$, we know that the quantity $G = pF_p - F$ is constant when evaluated on an extremal. The source of the expression G is somewhat mysterious. We did relate it to the law of conservation of energy, but that may not have satisfied you. This exercise will explore another way in which the expression G might have been discovered.
   (a) Interchange the roles of the independent and dependent variables in the integral in $(*)$ by making the substitution $y = u(x)$. If you let $v = u^{-1}$, then $x = v(y)$. Show that the functional in $(*)$ can be written as

$$\mathcal{F}(u) = \mathcal{H}(v) = \int_{a'}^{b'} H(y,v(y),v'(y))\,dy,$$

   where $u(a) = a'$, $u(b) = b'$, and $H(y,v,q) = qF(y,1/q)$.
   (b) You will notice that $H_v = 0$. Use this to show that G is constant.
3. A sphere of radius R can be paramerterized using the longitude θ and the polar angle ϕ (the compliment of the latitude) as parameters. The equations are

$$x = R\cos\theta\sin\phi, \quad y = R\sin\theta\sin\phi, \quad \text{and} \quad z = R\cos\phi. \qquad (**)$$

   (a) Show that the element of arc length on the sphere is given by $ds^2 = R^2[d\phi^2 + \sin^2\phi\,d\theta^2]$.
   (b) If we limit ourselves to curves which can be parameterized by the polar angle ϕ, with $\theta = w(\phi)$, $a \le \phi \le b$, show that

the length of the curve γ is given by

$$\mathcal{F}(w) = \int_\gamma ds = \int_a^b F(\phi, w, w')\, d\phi,$$

where $F(\phi, w, p) = R\sqrt{1 + p^2 \sin^2 \phi}$.

(c) Integrate the Euler equation for this functional and use the parametric equations for the sphere $(**)$ to show that there are constants A, B, and C such that the extremal curve lies in the plane through the origin defined by $Ax + By + Cz = 0$. Hence you will have proved that the geodesics on the sphere are great circles. The needed integral is hard to evaluate. Use a computer algebra system or a table of integrals, or come to see me.

4. Problem #19 on page 52 in Gelfand & Fomin.
5. Problem #20 on page 53 in Gelfand & Fomin.
6. When you use Lagrange multipliers to find the extremals of $\mathcal{F}(u)$ subject to the isoperimetric constraint $\mathcal{G}(u) = c$, what does it mean if the multiplier $\lambda = 0$?
7. Suppose that u is an extremal for the functional $\mathcal{F}(u)$ subject to an isoperimetric constraint of the form $\mathcal{G}(u) = c_1$. Show that *usually* u is an extremal for $\mathcal{G}(u)$ subject to a constraint of the form $\mathcal{F}(u) = c_2$. Explain carefully the condition that must be satisfied for this to be true.