

Homework #4

Due February 17.

Read Section 12 in Gelfand & Fomin.

1. Let S be a surface of revolution about the z -axis. It is defined by an equation of the form $x^2 + y^2 = f(z)$. Let γ be a geodesic on S and let $u(t) = (x(t), y(t), z(t))$ be its parameterization by arc length.
 - (a) Derive the Euler-Lagrange equations for x , y , and z .
 - (b) Show that $xy' - yx' = c_1$, where c_1 is a constant.
 - (c) Cylindrical coordinates r , θ , and z are related to the usual cartesian coordinates x , y , and z by the equations $x = r \cos \theta$ and $y = r \sin \theta$. Show that the element of arc length in \mathbf{R}^3 is given by $ds^2 = dr^2 + r^2 d\theta^2 + dz^2$.
 - (d) Express the parameterization for the geodesic γ in cylindrical coordinates $u(t) = (r(t), \theta(t), z(t))$, and find the Euler-Lagrange equations for r , θ , and z .
 - (e) Show that $r^2 \theta' = c_2$. Do this using the Euler equations for u . Then show that $c_2 = c_1$ where c_1 is the constant in part (a).
2. Consider the cylinder S in \mathbf{R}^3 defined by the equation $x^2 + y^2 = a^2$.
 - (a) The points $A = (a, 0, 0)$ and $B = (a \cos \alpha, a \sin \alpha, b)$ both lie on S . Find the geodesics joining them.
 - (b) Find 2 different extremals of the length functional joining $A = (a, 0, 0)$ and $C = (a, 0, 2\pi)$. How many extremals join A and C ?
3. Consider a surface S in \mathbf{R}^3 defined by

$$g(x, y, z) = 0. \tag{1}$$

A thread has its end fixed at the points P_1 and P_2 on S , and its position is given by the parameterization

$$u(t) = (x(t), y(t), z(t)) \quad 0 \leq t \leq L.$$

Assume that this is the parameterization by arc length, so

$$|u'(t)| = 1. \tag{2}$$

The thread has assumed an equilibrium position under the force of gravity (which points in the negative z direction). This means

that the potential energy is minimized, which is equivalent to

$$\int_0^L z(t) dt \quad (3)$$

being minimized. Describe how you would proceed to find u . In particular show the differential equations that u must satisfy. It is not necessary to solve the equations.

4. Suppose that $a < b$. Consider functions $u(x)$ defined on $[a, b]$ which satisfy $u(a) = u(b) = 0$, and $u(x) > 0$ for $a < x < b$. If we rotate the graph of u about the x -axis, we get a surface of revolution that is the boundary of a solid of revolution. Among all of those functions for which the area of the surface of revolution is equal to a fixed number A , we want to find the one for which the volume of the solid is maximum.

(a) Show that there are constants λ and C for which

$$u(x)^2 \sqrt{1 + u'(x)^2} + \lambda u(x) = C.$$

- (b) Use the boundary conditions on u to show that $C = 0$.
 (c) Find u explicitly, and describe the solid.