Homework #5

Due February 24.

Read Sections 12, 14, and 15 in Gelfand & Fomin.

1. Find the extremals of
$$\mathcal{F}(u) = \int_0^1 (a + u''(x)^2)\, dx$$
with end conditions $u(0) = 0$, $u'(0) = 1$, $u(1) = 1$, and $u'(1) = 1$.
2. Find the shortest curve(s) $y = u(x)$ for $a \le x \le b$ with the endpoints $u(a) = c$ and $u(b) = d$ unspecified. Solve the problem without doing any computations. You may and should cite known facts about the calculus of variations.
3. Find the shortest curve(s) in the planes connecting points on two dispoint circles. Solve the problem without doing any computations. You may and should cite known facts about circles and the calculus of variations.
4. Find the distance in the plane between the line $y = x$ and the parabola $y = 9/4 + x^2$. It is not necessary to reprove known facts. Use what you know about the calculus of variations to minimize computations.
5. Find the Euler equation that must be satisfied by an extremal of the functional
$$\mathcal{F}(u) = \int_a^b F(x, u(x), u'(x), u''(x), u'''(x))\, dx.$$
6. (#4 p. 64 in Gelfand & Fomin) Find the extremals of the functional
$$\mathcal{F}(u) = \int_0^{\pi/4} \left(u(x)^2 - u'(x)^2\right)\, dx$$
with $u(0) = 0$ and $u(\pi/4)$ unrestricted.
7. (#5 p. 64 in Gelfand & Fomin) Find the extremals of the functional
$$\mathcal{F}(u) = \int_0^b \frac{1}{u(x)} \sqrt{1 + u'(x)^2}\, dx$$
with $u(0) = 0$ and $g(b, u(b)) = 0$ where
   (a) $g(x, y) = x - y - 5$.
   (b) $g(x, y) = (x - 9)^2 + y^2 - 9$.
8. (#6 on p. 64 in Gelfand & Fomin) Describe the curve connecting two circles in the plane along which a particle will fall in minimum time under the force of gravity. In addition, write down the

equations which must be solved to find the constants involved. It is not necessary to solve them.