Homework #6

Due March 10.

Read Sections 14, and 15 in Gelfand & Fomin.

1. Hilbert's theorem says: Suppose that $F(x,u,p)$ is continuously differentiable for $a \leq x \leq b$ and for all $(u,p) \in \mathbf{R^2}$. Suppose that F_p is continuously differentiable and that $F_{pp}(x,u,p) > 0$ for $a \leq x \leq b$ and for all $(u,p) \in \mathbf{R^2}$. If u is a weak, piecewise continuously differentiable, local extremal for $\mathcal{F}(u) = \int_a^b F(x,u(x),u'(x))\,dx$ then $u \in C^2[a,b]$.

   (a) Show that the piecewise continuously differentiable function
   $$u(x) = \begin{cases} x & \text{for } 0 \leq x \leq 1/2, \\ 1-x & \text{for } 1/2 \leq x \leq 1, \end{cases}$$
   is a global minimum for the functional
   $$\mathcal{F}(u) = \int_0^1 \left(u'(x)^2 - 1\right)^2\,dx.$$

   (b) Why is this example not a counterexample to Hilbert's theorem?

2. (# 15 on p. 65 in Gelfand & Fomin) Show that the functional
$$\mathcal{F}(u) = \int_a^b \left(Au'(x)^2 + Bu(x)u'(x) + Cu(x)^2 + Du'(x) + Eu(x)\right)\,dx,$$
   where A, B, C, D, and E are constants and $A \neq 0$ has no broken extremals.
3. (# 16 on p. 65 in Gelfand & Fomin)
4. (# 17 on p. 65 in Gelfand & Fomin)
5. Can a local PC^1 extremum of $\int u'(x)^3\,dx$ have corners?
6. Consider the functional
$$\mathcal{F}(u) = \int_{-1}^1 u(x)^2\,(1 - u'(x))^2\,dx. \qquad (*)$$
   You might remember that we talked about this functional in class.

   (a) Find extremals of $\mathcal{F}$ with $u(-1) = 5$ and $u(1) = 3$.

   (b) Find extremals of $\mathcal{F}$ with $u(-1) = 5$ and $u(1) = 1$.

7. Again we consider the variational integral in $(*)$.
   (a) Find a candidate $u \in PC^1[-1,1]$ for a local minimum of $\mathcal{F}$ with
   $$u(-1) = -1/2 \quad \text{and} \quad u(1) = 1/2.$$
   (*Hint:* Since $u(-1) < 0$ and $u(1) > 0$, there is a point $x_0 \in (-1,1)$ where $u(x_0) = 0$.)
   (b) Find a candidate $u \in PC^1[-1,1]$ for a local minimum of $\mathcal{F}$ with
   $$u(-1) = 1 \quad \text{and} \quad u(1) = 1.$$