Homework #7

Due March 17.

Read Sections 16, 17, 18, and 19 in Gelfand & Fomin.

1. Consider $f(x) = e^x$ for $x \in \mathbf{R}$.
   (a) Show that f is convex and compute the Legendre transform $f^*(\xi)$.
   (b) Prove that $e^n x \le e^x + (n-1)e^n$ for all $x \in \mathbf{R}$, and all positive integers n.
2. Consider $f(x, y) = e^{(x^2+y^2)/2}$ for $(x, y) \in \Omega$, where

$$\Omega = \left\{ (x, y) \in \mathbf{R}^2 \mid \ |x| < 1 \text{ and } |y| < 1 \right\}.$$

   (a) Show that the Hessian matrix

$$Hf(x, y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

   is a positive definite matrix for all $(x, y) \in \Omega$. (*Remark:* There are formulaic methods for showing that a matrix is positive definite. If you are unfamiliar with them and cannot find them, please check with me or the grader.)
   (b) Let $\phi(x, y) = \nabla f(x, y)$. Compute and sketch $\Omega^* = \phi(\Omega)$. Show that Ω^* is not convex.
   (c) Show that the function $\alpha(t) = te^t$ is an increasing function on $[0, \infty)$ and maps $[0, \infty)$ onto itself. Let $\beta(s)$ be the inverse of α, so that $s = te^t$ if and only if $t = \beta(s)$. Show that

$$f^*(\xi, \eta) = [\beta(\xi^2 + \eta^2) - 1]e^{\beta(\xi^2+\eta^2)/2}.$$

   (*Remark:* If you show that

$$f^*(\xi, \eta) = (\xi^2 + \eta^2)e^{-\beta(\xi^2+\eta^2)/2} - e^{\beta(\xi^2+\eta^2)/2}$$

   you will be correct. Why?)
3. Consider the functional with Lagrangian $F(u, p) = \frac{1}{2}(p + ku)^2$, where k is a nonzero constant.
   (a) Find the canonical coordinates and the Hamiltonian.
   (b) Find the extremals of the functional by solving the Euler equations in their Hamiltonian form.
4. Consider the functional with Lagrangian $F(u, p) = p^2/2 - up$.
   (a) Find the canonical coordinates and the Hamiltonian.
   (b) Find the extremals of the functional by solving the Euler equations in their Hamiltonian form.

5. A vibrating spring with mass m moves in a straight line with spring equilibrium at $x = 0$, under the influence of a restoring force $F = -kx$.
   (a) Show that the Lagrangian is $F(x, p) = \frac{1}{2}p^2 - \frac{1}{2}kx^2$.
   (b) Find the canonical coordinates and the Hamiltonian, and provide a physical interpretation.
   (c) Find the extremals.