Homework #8

Due March 24.

Read Sections 18, and 19 in Gelfand & Fomin.

Consider the Hamiltonian system for the harmonic oscillator. The Hamiltonian is

$$H(x,\pi) = \frac{\pi^2}{2m} + \frac{kx^2}{2}.$$

The corresponding Hamiltonin system of differential equations is

$$\text{(H)} \qquad \begin{aligned} \frac{dx}{dt} &= \frac{\partial H}{\partial \pi} = \frac{\pi}{m}. \\ \frac{d\pi}{dt} &= -\frac{\partial H}{\partial x} = -kx. \end{aligned}$$

1. Show that under the transformation $x^* = x + \pi$, $\pi^* = \pi$ the system (H) of differential equations is transformed into
$$\begin{aligned} \frac{dx^*}{dt} &= \left(\frac{1}{m} + k\right)\pi^* - kx^* \\ \frac{d\pi^*}{dt} &= k(\pi^* - x^*) \end{aligned}$$
2. Show that the transformation $x^* = x + \pi$, $\pi^* = \pi$ is a canonical transformation.
3. Show that under the transformation $x^* = e^x$, $\pi^* = \pi$ the system (H) of differential equations is transformed into
$$\begin{aligned} \frac{dx^*}{dt} &= \frac{\pi^* x^*}{m} \\ \frac{d\pi^*}{dt} &= -k \log x^* \end{aligned}$$
4. Show that the transformation $x^* = e^x$, $\pi^* = \pi$ is NOT a canonical transformation. (*Hint:* Show that the system in Exercise 3 is not a Hamiltonian system.)
5. Find a function $\pi^* = \pi^*(x,\pi)$ such that $x^* = e^x$, $\pi^* = \pi^*(x,\pi)$ is a canonical transformation.

The method we used to approach canonical transformations makes it look as though a canonical transformation depends on a Hamiltonian. This is not true if everything is independent of the independent variable x.

6. Show that the transformation

$$u^* = u^*(u, \pi)$$
$$\pi^* = \pi^*(x, \pi)$$

is a canonical transformation if there is a function G such that

$$dG = \sum_{j=1}^{N} \pi_j^* \, du_j^* - \sum_{j=1}^{N} \pi_j \, du_j.$$

7. Show that in $\mathbf{R}^2$ the transformation

$$u^* = \sqrt{u}\cos(2\pi)$$
$$\pi^* = \sqrt{u}\sin(2\pi)$$

is canonical.