

Homework #10

Due April 9.

Read Sections 22, 23, 24, and 25 in Gelfand & Fomin.

1. Consider the functional with Lagrangian

$$F(x, u, p) = \sqrt{x^2 + u^2} \sqrt{1 + p^2}.$$

- (a) Find the Hamiltonian and the associated Hamiltonian system.
- (b) Show that $u^2 - \pi^2$ is a conserved quantity.
- (c) Find a solution to the Hamilton-Jacobi equation with the form

$$S(x, u) = \frac{1}{2} (Ax^2 + 2Bxu + Cu^2).$$

- (d) Show that the solutions are given implicitly in the form

$$(x^2 - u^2) \cos \alpha + 2ux \sin \alpha = C,$$

where α and C are constants.

2. Consider a particle with mass m in \mathbf{R}^3 moving under the influence of the potential V . Of course, the Lagrangian is

$$L(x, y, z, p, q, r) = \frac{m}{2} (p^2 + q^2 + r^2) - V(x, y, z).$$

- (a) Write the Lagrangian in polar coordinates r , θ , and ϕ , defined by $x = r \sin \phi \cos \theta$, $y = r \sin \phi \sin \theta$, and $z = r \cos \phi$.
- (b) What is the corresponding Hamiltonian?
- (c) Assuming that $V = -k/r$, reduce the solution of the Hamilton-Jacobi equation to the evaluation of integrals. (It is not necessary to evaluate the integrals.)

3. Calculate the second variation of the functional

$$\mathcal{F}(u) = \int_a^b F(x, u(x), u'(x)) dx$$

under the following assumptions.

- (a) $F(x, u, p)$ does not depend on u .
 - (b) $F(x, u, p)$ does not depend on p .
4. Suppose $F(x, u, p) = A(x)u + B(x)p$, where A and B are functions depending only on x . Compute the first variation $\delta\mathcal{F}(u, \phi)$, and the second variation $\delta^2\mathcal{F}(u, \phi)$.
 5. Suppose that $F(x, u, p) = A(x) + B(x)u^2 + 2C(x)up + D(x)p^2$. Compute the first and second variations.

Final Exam: The final exam will be made available on the last day of class, April 17, and it will be due on the last day of exams, April 29. To help me write the exam, I would appreciate it if you would submit a problem or two that you think would be appropriate for the exam. I will distribute all such problems to the class as a whole. However, I will maintain my prerogative to use all, some, or none of them.

The deadline for submissions is Tuesday, April 10.

There is no obligation to submit anything, and there will be no credit or discredit attached to any submission or lack thereof.