Homework #11

Due April 16.

Read Sections 25, 26, 27, and 28 in Gelfand & Fomin.

Since what I am doing in class differs from what is in the book, I will give a brief outline of the subject as I covered it.

We are looking at the functional

$$\mathcal{F}(u) = \int_a^b F(x, u(x), u'(x) \quad \text{with} \tag{1}$$

$$u(a) = c \quad \text{and} \quad u(b) = d. \tag{2}$$

where F is continuously differentiable. We made the following definitions.

Definition 1. The function $u \in C^1$ is a *weak mimimizer* for $\mathcal{F}$ if there is an $\epsilon > 0$ such that $\mathcal{F}(v) \geq \mathcal{F}(u)$ for all $v \in C^1$ which satisfy the boundary conditions (**??**) and $||v - u||_1 < \epsilon$.

Definition 2. The function $u \in C^1$ is a *strong mimimizer* for $\mathcal{F}$ if there is an $\epsilon > 0$ such that $\mathcal{F}(v) \geq \mathcal{F}(u)$ for all $v \in C^1$ which satisfy the boundary conditions (**??**) and $||v - u||_0 < \epsilon$.

We were able to prove the next result.

Theorem 3.

1. *If u is a weak minimizer for $\mathcal{F}$, then*

   $$\delta\mathcal{F}(u, \phi) = 0 \quad \textit{and} \quad \delta^2\mathcal{F}(u, \phi) \geq 0 \quad \textit{for all } \phi \in C_0^1(a, b).$$

2. *If $u \in C^1[a, b]$ is a weak extremal for $\mathcal{F}$ and there is $k > 0$ suxh that*

   $$\delta^2\mathcal{F}(u, \phi) \geq k \int_a^b \left[\phi(x)^2 + \phi'(x)^2\right] dx$$

   for all $\phi \in C_0^1(a, b)$ then u is a strict weak mimimizer for $\mathcal{F}$.

A strong minimizer is always a weak minimizer, but this example shows that the opposite is not always true.

Example 4. Consider the funtional $\mathcal{F}(u) = \int_0^1 [u'(x)^2 + u'(x)^3]\, dx$ with $u(0) = 0$ and $u(1) = 0$. The function $u(x) = 0$ is a strict weak minimizer, but not a strong minimizer.

You may remember that we used this inequality to help us with the previous example.

Lemma 5 (Poincaré Inequality). *Suppose that $f \in C^1[a,b]$ and $f(0) = 0$. Then*

$$\int_a^b f(x)^2\,dx \le (b-a)^2 \int_a^b f'(x)^2\,dx.$$

The next example shows that the condition on the second variation in part 2 of Theorem 3 cannot be weakened very much.

Example 6. Consider the functional $\mathcal{F}(u) = \int_{-1}^1 \left[x^2 u'(x)^2 + x u'(x)^3\right]\,dx$ with $u(-1) = u(1) = 0$. The function $u(x) = 0$ is a weak extremal, and $\delta^2\mathcal{F}(u,\phi) > 0$ for all $\phi \in C_0^1(a,b)$, but u is not a weak minimizer.

However, we also have the following result, which at first glance may seem to contradict Example 6.

Theorem 7. *Suppose $\mathcal{F}$ is a functional for which*

$$\delta^2\mathcal{F}(u,\phi) \ge 0, \quad \textit{for all } w \in C^1[a,b] \textit{ and for all } \phi \in c_0^1(a,b).$$

If $u \in C_0^1[a,b]$ is a weak extremal for $\mathcal{F}$, then u is a global minimizer for $\mathcal{F}$. In other words,

$$\mathcal{F}(v) \ge \mathcal{F}(u) \quad \textit{for all } v \in C^1[a,b].$$

Now on to the homework assignment. One bit of semi-bad news. **By University rules, no lates can be allowed on this assignment.**

1. Consider the funtional $\mathcal{F}(u) = \int_0^1 \left[u'(x)^2 + u'(x)^3\right]\,dx$ with $u(0) = 0$ and $u(1) = c$.
   (a) Show that $u(x) = cx$ is the only extremal.
   (b) For which values of c is Legendre's condition satisfied?
   (c) Show that for $c > -1/3$, u is a strict weak minimizer.
   (d) Show that for $c < -1/3$, u is not a weak minimizer.
2. The Lagrangian for the problem of finding graphs of minimum length in $\mathbf{R}^2$ is $F(p) = \sqrt{1+p^2}$. Show that the extremals are actually strict weak minimizers.
3. Show that the extremals in Problem 2 are actually global minimzers. (If you do this problem, it is not necessary to do Problem 2.)
4. Consider the functional $\mathcal{F}(u) = \int_0^1 u'(x)^3\,dx$ with $u(0) = 2$ and $u(1) = 3$.
   (a) Find the unique extremal.
   (b) Is this extremal a weak minimizer? Is it a strict minimizer?
   (c) Repeat parts (a) and (b) with $u(1) = 1$.

5. Consider the functional $\mathcal{F}(u) = \int_0^1 \left[u'(x)^2 - 2u(x)^2 + 2xu(x)\right] dx$ with $u(0) = 0$ and $u(1) = 1/4$.
   (a) Find the unique extremal.
   (b) Is this extremal a weak minimizer? Is it a strict minimizer?

Final Exam: The final exam will be made available on the last day of class, April 16, and it will be due on the last day of exams, April 29. It will be an open book, open notes, take home exam with a five hour time limit.