Homework #11

Due April 16.

Read Sections 25, 26, 27, and 28 in Gelfand & Fomin.

Since what I am doing in class differs from what is in the book, I will give a brief outline of the subject as I covered it.

We are looking at the functional

$$\mathcal{F}(u) = \int_a^b F(x, u(x), u'(x) \quad \text{with} \tag{1}$$

$$u(a) = c \quad \text{and} \quad u(b) = d. \tag{2}$$

where F is continuously differentiable. We made the following definitions.

Definition 1. The function $u \in C^1$ is a *weak mimimizer* for $\mathcal{F}$ if there is an $\epsilon > 0$ such that $\mathcal{F}(v) \geq \mathcal{F}(u)$ for all $v \in C^1$ which satisfy the boundary conditions (2) and $||v - u||_1 < \epsilon$.

Definition 2. The function $u \in C^1$ is a *strong mimimizer* for $\mathcal{F}$ if there is an $\epsilon > 0$ such that $\mathcal{F}(v) \geq \mathcal{F}(u)$ for all $v \in C^1$ which satisfy the boundary conditions (2) and $||v - u||_0 < \epsilon$.

We were able to prove the next result.

Theorem 3.

1. *If u is a weak minimizer for $\mathcal{F}$, then*

   $$\delta\mathcal{F}(u, \phi) = 0 \quad \textit{and} \quad \delta^2\mathcal{F}(u, \phi) \geq 0 \quad \textit{for all } \phi \in C_0^1(a, b).$$

2. *If $u \in C^1[a, b]$ is a weak extremal for $\mathcal{F}$ and there is $k > 0$ suxh that*

   $$\delta^2\mathcal{F}(u, \phi) \geq k \int_a^b \left[\phi(x)^2 + \phi'(x)^2\right] dx$$

   for all $\phi \in C_0^1(a, b)$ then u is a strict weak mimimizer for $\mathcal{F}$.

A strong minimizer is always a weak minimizer, but this example shows that the opposite is not always true.

Example 4. Consider the funtional $\mathcal{F}(u) = \int_0^1 [u'(x)^2 + u'(x)^3]\, dx$ with $u(0) = 0$ and $u(1) = 0$. The function $u(x) = 0$ is a strict weak minimizer, but not a strong minimizer.

You may remember that we used this inequality to help us with the previous example.

Lemma 5 (Poincaré Inequality)**.** *Suppose that $f \in C^1[a,b]$ and $f(0) = 0$. Then*

$$\int_a^b f(x)^2\,dx \le (b-a)^2 \int_a^b f'(x)^2\,dx.$$

The next example shows that the condition on the second variation in part 2 of Theorem 3 cannot be weakened very much.

Example 6. Consider the functional $\mathcal{F}(u) = \int_{-1}^{1} \left[x^2 u'(x)^2 + x u'(x)^3\right] dx$ with $u(-1) = u(1) = 0$. The function $u(x) = 0$ is a weak extremal, and $\delta^2\mathcal{F}(u,\phi) > 0$ for all $\phi \in C_0^1(a,b)$, but u is not a weak minimizer.

However, we also have the following result, which at first glance may seem to contradict Example 6.

Theorem 7. *Suppose $\mathcal{F}$ is a functional for which*

$$\delta^2\mathcal{F}(u,\phi) \ge 0, \quad \text{for all } w \in C^1[a,b] \text{ and for all } \phi \in c_0^1(a,b).$$

If $u \in C_0^1[a,b]$ is a weak extremal for $\mathcal{F}$, then u is a global minimizer for $\mathcal{F}$. In other words,

$$\mathcal{F}(v) \ge \mathcal{F}(u) \quad \text{for all } v \in C^1[a,b].$$

Now on to the homework assignment. One bit of semi-bad news.

1. Consider the funtional $\mathcal{F}(u) = \int_0^1 \left[u'(x)^2 + u'(x)^3\right] dx$ with $u(0) = 0$ and $u(1) = c$.
   (a) Show that $u(x) = cx$ is the only extremal.
   (b) For which values of c is Legendre's condition satisfied?
   (c) Show that for $c > -1/3$, u is a strict weak minimizer.
   (d) Show that for $c < -1/3$, u is not a weak minimizer.

   Answers:
   (a) An extremal must satisfy the Euler equation $F_u = \frac{d}{dx}F_p$. In this case $F_u = 0$ so we have $F_p =$ constant. Since $F_p = 2p + 3p^2$, we see that p is a constant. Hence $u'(x) = C$ and $u(x) = Cx + B$. The boundary conditions imply that $B = 0$ and $C = c$. Hence $u(x) = cx$.
   (b) Legendre's condition says that $F_{pp} \ge 0$. In this case $F_{pp} = 2 + 6p = 2 + 6c$ so we must have $c \ge -1/3$.
   (c) The conditions that imply that $u = cx$ is a strict weak minimizer are:
      - u is an extremal. This is true for all c.
      - The strict Legendre condition: $F_{pp} > 0$. We need $2 + 6c > 0$ or $c > -1/3$.

- The strict Jacobi condition: There are no points conjugate to 0 in $(0,1]$. In this case the Jacobi equation is $-\left((2+6c)v'\right)' = 0$, or $v'' = 0$. The Jacobi fields with $v(0) = 0$ are $v(x) = Cx$, where C is a non-zero constant. Hence there are no points conjugate to 0 in the whole real line.

Hence the limiting condition for $u = cx$ to be a strict weak minimizer is $c > -1/3$.

(d) The Legendre condition is necessary for a weak minimizer. Hence by part (a) we need $c \geq -1/3$. Hence if $c < -1/3$ $u(x) = cx$ is not a weak minimizer.

2. The Lagrangian for the problem of finding graphs of minimum length in $\mathbf{R}^2$ is $F(p) = \sqrt{1+p^2}$. Show that the extremals are actually strict weak minimizers.

Answer: We know that the extremals are the affine functions $u(x) = Ax+B$. We compute that $F_{pp} = \frac{1}{(1+p^2)^{3/2}} > 0$. The Jacobi fields that vanish at a have the form $v(x) = C(x-a)$, where C is a constant. Hence there are no points in the real line conjugate to a. Consequently, every extremal is a strict weak minimizer.

3. Show that the extremals in Problem 2 are actually global minimizers. (If you do this problem, it is not necessary to do Problem 2.)

Answer: We have $F_{pp} = 1/(1+p^2)^{3/2}$ and $F_{up} = F_{uu} = 0$, so for all $\phi \in C^1_0(a,b)$, and $w \in C^1[a,b]$ we have

$$\delta^2\mathcal{F}(u,\phi) = \int_a^b \frac{\phi'(x)^2}{(1+w'(x)^2)^{3/2}}\,dx \geq 0.$$

Hence by Theorem 7 above, any extremal is a global minimizer.

4. Consider the functional $\mathcal{F}(u) = \int_0^1 u'(x)^3\,dx$ with $u(0) = 2$ and $u(1) = 3$.
   (a) Find the unique extremal.
   (b) Is this extremal a weak minimizer? Is it a strict minimizer?
   (c) Repeat parts (a) and (b) with $u(1) = 1$.

   Answers:

   (a) We have $F = p^3$. Since $F_u = 0$, Euler's equation implies that $F_p = 3p^2$ is constant, so $u'(x)$ is constant if u is an extremal. Hence the extremals are the affine functions $u(x) = Ax+B$. The boundary conditions imply that $u(x) = 2+x$.

   (b) Since $F_{pp} = 6p$, the strict Legendre condition is satisfied if and only if $p > 0$. For the extremal $u(x) = 2+x$ this is satisfied. The Jacobi equation is $-v'' = 0$, so the Jacobi fields with $v(0) = 0$ are of the form $v(x) = Cx$, so there are no points in the real line conjugate to 0. Hence $u(x) = 2+x$ is a strict weak minimizer.

(c) In this case the extremal is $u(x) = 2-x$. Since $u'(x) = -1$, Legendre's condition is not satisfied, so this is not a weak extremal.

5. Consider the functional $\mathcal{F}(u) = \int_0^1 [u'(x)^2 - 2u(x)^2 + 2xu(x)]\, dx$ with $u(0) = 0$ and $u(1) = 1/4$.
   (a) Find the unique extremal.
   (b) Is this extremal a weak minimizer? Is it a strict minimizer?

Answers:

(a) The Euler equation is $u'' + 2u = x$. The general solution is

$$u(x) = \frac{x}{2} + A\sin(x\sqrt{2}) + B\cos(x\sqrt{2}),$$

So these are the extremals. In the case at hand, the boundary conditions imply that $B = 0$ and $A = -1/4\sin(1/\sqrt{2})$, so

$$u(x) = \frac{x}{2} - \frac{\sin(x\sqrt{2})}{4\sin(\sqrt{2})}.$$

(b) We have $F_{pp} = 2 > 0$, so the strict Legendre condition is satisfied. In addition, $F_{up} = 0$ and $F_{uu} = -4$. Hence the Jacobi equation is $-(2v')' - 4v = 0$, or $v'' + 2v = 0$. The solutions which satisfy $v(0) = 0$ are of the form $v(x) = C\sin(x\sqrt{2})$. The smallest positive zero of this function is $x_0 = \pi/\sqrt{2}$. Since $1 < x_0$, the strict Jacobi condition is satisfied. Therefore the extremal in part (a) is a strict weak extremal.