Notes on Noether's Theorem

John C. Polking

Introduction

Noether's Theorem says that if a functional is invariant under a family of transformations, then there is a conservation law that must be satisfied by any extremal. In these notes we will state and prove this result.

1. Invariance of a functional under a transformation

We will be considering a functional

$$\mathcal{F}(u) = \int_a^b F(x, u(x), u'(x))\, dx. \tag{1.1}$$

Since we will be changing the interval of integration, we will set $I = (a, b)$ and indicate the dependence on I by writing (1.1) as

$$\mathcal{F}(u, I) = \int_a^b F(x, u(x), u'(x))\, dx. \tag{1.2}$$

We will be interested in how the functional changes under a change of variables

$$y = X(x, u, p) \tag{1.3}$$

$$v = U(x, u, p) \tag{1.4}$$

Making this coordinate change in (1.2) is a two step process.

- Substitute $u = u(x)$ and $p = u'(x)$ in (1.3). We get a function

$$\eta(x) = y = X(x, u(x), u'(x)). \tag{1.5}$$

  We will assume that η is invertible and maps the interval I onto a new interval $I^* = (a^*, b^*)$. Denote the inverse of η by ξ, so that

$$\eta(x) = y \quad \text{if and only if} \quad \xi(y) = x. \tag{1.6}$$

- Substitute $x = \xi(y)$, $u = u(\xi(y))$, and $p = u'(\xi(y))$ in (1.4) to get a function

$$v(y) = U(\xi(y), u(\xi(y)), u'(\xi(y))). \tag{1.7}$$

  Then $v : I^* \to \mathbf{R}^N$, just like $u : I \to \mathbf{R}^N$.

Definition 1. The functional $\mathcal{F}$ is *invariant* under the transformation in (1.3) and (1.4) if

$$\mathcal{F}(u, I) = \mathcal{F}(v, I^*) \tag{1.8}$$

for all intervals I and for all functions $u : I \to \mathbf{R}^N$.

If we write out the formulas, equation (1.8) becomes

$$\int_a^b F\left(x, u(x), \frac{du}{dx}(x)\right)\, dx = \int_{a^*}^{b^*} F\left(y, v(y), \frac{dv}{dy}(y)\right)\, dy. \tag{1.9}$$

Example 1. Suppose the Lagrangian $F = F(u, p)$ is independent of the independent variable x. Then $\mathcal{F}(u, I) = \int_a^b F(u(x), u'(x))\, dx$. The transformation we will consider is a translation in the independent variable x, so

$$y = x + h \quad \text{and} \quad v = u.$$

Then $\eta(x) = x + h$, its inverse is $\xi(y) = y - h$, and $I^* = (a + h, b + h)$. Furthermore $v(y) = u(\xi(y)) = u(y - h)$. Hence,

$$\frac{dv}{dy}(y) = \frac{du}{dx}(y - h).$$

Therefore,

$$\begin{aligned}
\mathcal{F}(v, I^*) &= \int_{a+h}^{b+h} F(v(y), v'(y))\, dy \\
&= \int_{a+h}^{b+h} F(u(y - h), u'(y - h))\, dy
\end{aligned}$$

When we make the change of variables $y = x + h$, this becomes

$$\mathcal{F}(v, I^*) = \int_a^b F(u(x), u'(x))\, dx = \mathcal{F}(u, I).$$

Thus we see that this functional is invariant under translations in the independent variable.

Example 2. Next let's consider the functional with Lagrangian $F = x|p|^2$, and the same transformation. Now the same computation gives us

$$\begin{aligned}
\mathcal{F}(v, I^*) &= \int_{a+h}^{b+h} y \left|\frac{dv}{dy}(y)\right|^2\, dy \\
&= \int_{a+h}^{b+h} y \left|\frac{du}{dx}(y - h)\right|^2\, dy \\
&= \int_a^b (x + h) \left|\frac{du}{dx}(x)\right|^2\, dx \\
&= \mathcal{F}(u, I) + h \int_a^b \left|\frac{du}{dx}(x)\right|^2\, dx.
\end{aligned}$$

Consequently, this functional is not invariant under the translation.

2. Invariance under a family of transformations

Suppose we have a family of transformations

$$y = X(x, u, p, \epsilon) \quad \text{and} \quad v = U(x, u, p, \epsilon) \tag{2.1}$$

defined for all ϵ satisfying $|\epsilon| < \epsilon_0$ for some $\epsilon_0 > 0$. We will assume that this family is a variation of the identity in the sense that at $\epsilon = 0$ we have

$$X(x, u, p, 0) = x \quad \text{and} \quad U(x, u, p, 0) = u. \tag{2.2}$$

Let's examine what it means that a functional $\mathcal{F}$ is invariant under all of these transformations.

Suppose we have a function $u : I \to \mathbf{R}^N$. Following the agenda in Section 1, we define

$$\eta(x, \epsilon) = X(x, u(x), u'(x), \epsilon). \tag{2.3}$$

Because of (2.2) we have

$$\eta(x, 0) = x. \tag{2.4}$$

Thus, $\eta_x(x, 0) = 1$, so for small $|\epsilon|$ we must have $\eta_x(x, \epsilon) > 0$ for all $x \in I$. this means that $\eta(\,\cdot\,, \epsilon)$ is an increasing function and therefore $\eta(\,\cdot\,, \epsilon)$ isinvertible, mapping I onto a new interval $I_\epsilon = (a_\epsilon, b_\epsilon)$. Let $\xi(y, \epsilon)$ denote the inverse of η, so that

$$\eta(x, \epsilon) = y \quad \text{if and only if} \quad \xi(y, \epsilon) = x. \tag{2.5}$$

In agreement with (1.7), we set

$$v(y, \epsilon) = U(\xi(y, \epsilon), u(\xi(y, \epsilon)), u'(\xi(y, \epsilon)), \epsilon). \tag{2.6}$$

Notice that, because of (2.2) and (2.4), $\xi(y, 0) = y$, and

$$v(y, 0) = u(y). \tag{2.7}$$

Thus the family $v(\,\cdot\,, \epsilon)$ is a variation of the function u.

Finally, the fact that the functional $\mathcal{F}$ is invariant under the family of transformations in (2.1) means that

$$\mathcal{F}(v(\,\cdot\,, \epsilon), I_\epsilon) = \mathcal{F}(u, I) \quad \text{for all small } \epsilon.$$

If we define the function

$$\Phi(\epsilon) = \mathcal{F}(v(\,\cdot\,, \epsilon), I_\epsilon), \tag{2.8}$$

then $\Phi(\epsilon)$ is a constant function. In particular $\Phi'(\epsilon) = 0$. To prove Noether's Theorem we will only need that $\Phi'(0) = 0$. Let's give that assumption a name of its own.

Definition 2. The functional $\mathcal{F}$ is *infinitesimally invariant* under the family defined in (2.1) if the function Φ defined in (2.8) satisfies $\Phi'(0) = 0$.

3. Noether's Theorem

We are now ready to state the theorem.

THEOREM 3.1. *Suppose the functional* $\mathcal{F}(u) = \int_a^b F(x, u(x), u'(x))\, dx$ *is infinitesimally invariant under the family of transformations*

$$y = X(x, u, p, \epsilon) \quad \textit{and} \quad v = U(x, u, p, \epsilon),$$

where $X(x, u, p, 0) = x$ *and* $U(x, u, p, 0) = u$*. Then the quantity*

$$\mu \left(F - \sum_{j=1}^{N} p^j F_{p^j} \right) + \sum_{j=1}^{N} \phi^j F_{p^j} \tag{3.1}$$

is constant along any extremal of $\mathcal{F}$*, where*

$$\mu(x, u, p) = X_\epsilon(x, u, p, 0), \quad \textit{and} \tag{3.2}$$

$$\phi(x, u, p) = U_\epsilon(x, u, p, 0). \tag{3.3}$$

REMARK 3.1. Thus any family of transformations which leaves $\mathcal{F}$ invariant, or even infinitesimally invariant, leads to a conservation law for $\mathcal{F}$.

The proof involves a lot of straightforward computation. However, there is so much computation that it easy to get lost. We will proceed in a systematic manner.

Let u be an extremal for $\mathcal{F}$, which will be fixed throughout. Define

$$\overline{\mu}(x) = \mu(x, u(x), u'(x)), \quad \text{and} \tag{3.4}$$

$$\overline{\phi}(x) = \phi(x, u(x), u'(x)). \tag{3.5}$$

We have

$$\Phi(\epsilon) = \mathcal{F}(v(\,\cdot\,, \epsilon)) = \int_{a_\epsilon}^{b_\epsilon} F(y, v(y, \epsilon), v_y(y, \epsilon))\, dy.$$

To avoid having to deal with endpoints that depend on ϵ when we differentiate Φ, we substitute $y = \eta(x, \epsilon)$. Then $dy = \eta_x(x, \epsilon)\, dx$, so

$$\Phi(\epsilon) = \int_a^b F(\eta(x, \epsilon), v(\eta(x, \epsilon), \epsilon), v_y(\eta(x, \epsilon), \epsilon))\eta_x(x, \epsilon)\, dx. \tag{3.6}$$

The integrand in (3.6) is

$$f(x, \epsilon) = F(\eta(x, \epsilon), v(\eta(x, \epsilon), \epsilon), v_y(\eta(x, \epsilon), \epsilon))\eta_x(x, \epsilon). \tag{3.7}$$

Hence

$$\Phi(\epsilon) = \int_a^b f(x, \epsilon)\, dx \quad \text{and} \quad \Phi'(\epsilon) = \int_a^b f_\epsilon(x, \epsilon)\, dx.$$

First a lemma that tells us what we need to know about η and ξ.

LEMMA 3.2. *The function* $\eta(x, \epsilon)$*, defined in* (2.3)*, and its inverse* $\xi(y, \epsilon)$ *satisfy*

(1) $\eta(x, 0) = x$*, and* $\eta_x(x, 0) = 1$,
(2) $\eta_\epsilon(x, 0) = \overline{\mu}(x)$*, and* $\eta_{\epsilon x}(x, 0) = \overline{\mu}'(x)$,
(3) $\xi(y, 0) = y$*,* $\xi_y(y, 0) = 1$*, and* $\xi_{yy}(y, 0) = 0$,
(4) $\xi_\epsilon(y, 0) = -\overline{\mu}(y)$*, and* $\xi_{\epsilon y}(y, 0) = -\overline{\mu}'(y)$.

Proof. We need only prove the first equation of each part since the rest follows by differentiation. Part (1) is equation (2.4). Part (2) comes from differentiating (2.3) and using (3.4).

Part (3) follows from part (1) and the fact that ξ is the inverse of η. To prove (4), we differentiate the identity $\eta(\xi(y,\epsilon),\epsilon) = y$ with respect to ϵ, obtaining

$$\eta_x(\xi(y,\epsilon),\epsilon)\xi_\epsilon(y,\epsilon) + \eta_\epsilon(\xi(y,\epsilon),\epsilon) = 0.$$

Evaluating at $\epsilon = 0$, and using the previous parts we get the desired result. □

Turning to the calculation of $f_\epsilon(x,0)$, we notice that by the product rule for differentiation,

$$\begin{aligned}\frac{\partial f}{\partial \epsilon}(x,\epsilon) = \frac{\partial}{\partial \epsilon}\left[F(\eta(x,\epsilon), v(\eta(x,\epsilon),\epsilon), v_y(\eta(x,\epsilon),\epsilon))\right]\eta_x(x,\epsilon)\\ + F(\eta(x,\epsilon), v(\eta(x,\epsilon),\epsilon), v_y(\eta(x,\epsilon),\epsilon))\eta_{x\epsilon}(x,\epsilon).\end{aligned}$$

To make the formulas simpler we will use the notation $(\dots)$ to indicate the rather large set of variables in $F(\dots)$. Then using the chain rule and evaluating at $\epsilon = 0$ using Lemma 3.2, we get

$$\begin{aligned}\frac{\partial f}{\partial \epsilon}(x,0) &= F_x(\dots)\overline{\mu}(x)\\ &\quad + \sum_{j=1}^{N} F_{u^j}(\dots)\frac{\partial}{\partial \epsilon}\left[v^j(\eta(x,\epsilon),\epsilon)\right]_{\epsilon=0}\\ &\quad + \sum_{j=1}^{N} F_{p^j}(\dots)\frac{\partial}{\partial \epsilon}\left[v^j_y(\eta(x,\epsilon),\epsilon)\right]_{\epsilon=0}\\ &\quad + F(\dots)\overline{\mu}'(x).\end{aligned} \tag{3.8}$$

It remains to compute $\frac{\partial}{\partial\epsilon}\left[v(\eta(x,\epsilon),\epsilon)\right]_{\epsilon=0}$ and $\frac{\partial}{\partial\epsilon}\left[v_y(\eta(x,\epsilon),\epsilon)\right]_{\epsilon=0}$. We will do this in the next lemma.

Lemma 3.3. *We have*

$$\frac{\partial}{\partial \epsilon}\left[v(\eta(x,\epsilon),\epsilon)\right]_{\epsilon=0} = \overline{\phi}(x), \quad \textit{and} \tag{3.9}$$

$$\frac{\partial}{\partial \epsilon}\left[v_y(\eta(x,\epsilon),\epsilon)\right]_{\epsilon=0} = \overline{\phi}'(x) - \overline{\mu}'(x)u'(x). \tag{3.10}$$

Proof. From equation (2.6) we have that

$$v(y,\epsilon) = U(\xi(y,\epsilon), u(\xi(y,\epsilon)), u'(\xi(y,\epsilon)), \epsilon).$$

It will be convenient to split this by defining

$$w(x,\epsilon) = U(x, u(x), u'(x), \epsilon), \tag{3.11}$$

so that

$$v(y,\epsilon) = w(\xi(y,\epsilon),\epsilon). \tag{3.12}$$

Notice that

$$w(x,0) = U(x,u(x),u'(x),0) = u(x), \text{and} \tag{3.13}$$

$$w_\epsilon(x,0) = U_\epsilon(x,u(x),u'(x),0) = \overline{\phi}(x). \tag{3.14}$$

Since ξ is the inverse of η,

$$v(\eta(x,\epsilon),\epsilon) = w(\xi(\eta(x,\epsilon),\epsilon),\epsilon) = w(x,\epsilon).$$

Hence, using (3.14),

$$\frac{\partial}{\partial\epsilon}\left[v(\eta(x,\epsilon),\epsilon)\right]_{\epsilon=0} = w_\epsilon(x,0) = \overline{\phi}(x),$$

proving (3.9).

From (3.12), we compute that

$$v_y(y,\epsilon) = w_x(\xi(y,\epsilon),\epsilon)\xi_y(y,\epsilon).$$

Consequently,

$$\begin{aligned} v_y(\eta(x,\epsilon),\epsilon) &= w_x(\xi(\eta(x,\epsilon),\epsilon)\xi_y(\eta(x,\epsilon),\epsilon) \\ &= w_x(x,\epsilon)\xi_y(\eta(x,\epsilon),\epsilon). \end{aligned}$$

Differentiating with respect to ϵ, and evaluating at $\epsilon = 0$, we get

$$\begin{aligned} \frac{\partial}{\partial\epsilon}\left[v_y(\eta(x,\epsilon),\epsilon)\right]_{\epsilon=0} &= w_{x\epsilon}(x,0)\xi_y(x,0) \\ &\qquad + w_x(x,0)\frac{\partial}{\partial\epsilon}\left[\xi_y(\eta(x,\epsilon),\epsilon)\right]_{\epsilon=0}. \end{aligned}$$

Using Lemma 3.2 and equations (3.13), and (3.14), this becomes

$$\frac{\partial}{\partial\epsilon}\left[v_y(\eta(x,\epsilon),\epsilon)\right]_{\epsilon=0} = \overline{\phi}'(x) + u'(x)\frac{\partial}{\partial\epsilon}\left[\xi_y(\eta(x,\epsilon),\epsilon)\right]_{\epsilon=0}. \tag{3.15}$$

To evaluate the last term on the right in equation (3.15), we differentiate and use Lemma 3.2 to get

$$\begin{aligned} \frac{\partial}{\partial\epsilon}\left[\xi_y(\eta(x,\epsilon),\epsilon)\right]_{\epsilon=0} &= \xi_{yy}(x,0)\eta_\epsilon(x,0) + \xi_{\epsilon y}(x,0) \\ &= -\overline{\mu}'(x), \end{aligned}$$

Substituting this into equation (3.15) we get (3.10), finishing the proof of Lemma 3.3 □

Substituting the results of Lemma 3.3 into equation (3.8) yields

$$\begin{aligned} \frac{\partial f}{\partial\epsilon}(x,0) = F_x(\dots)\overline{\mu}(x) &+ \sum_{j=1}^N F_{u^j}(\dots)\overline{\phi}^j(x) \\ &+ \sum_{j=1}^N F_{p^j}(\dots)\left[\overline{\phi}^{j\prime}(x) - \overline{\mu}'(x)u^{j\prime}(x)(x)\right] \\ &+ F(\dots)\overline{\mu}'(x). \end{aligned}$$

Next we substitute this into $\Phi'(0) = \int_a^b f_\epsilon(x,0)\,dx$, in the process separating those terms involving $\overline{\mu}$ and $\overline{\phi}$. Thus,

$$\begin{aligned} 0 = \Phi'(0) = \sum_{j=1}^N &\int_a^b \left[\overline{\phi}^j F_{u^j} + \overline{\phi}^{j\prime} F_{p^j}\right]\,dx \\ &+ \int_a^b \left\{\overline{\mu}F_x + \overline{\mu}'\left[F - \sum_{j=1}^N u^{j\prime}F_{p^j}\right]\right\}\,dx. \end{aligned} \tag{3.16}$$

We integrate by parts in the first integral to obtain

$$\int_a^b \left[\overline{\phi}^j F_{u^j} + \overline{\phi}^{j\prime} F_{p^j}\right] dx = \int_a^b \overline{\phi}^j \left[F_{u^j} - \frac{d}{dx} F_{p^j}\right] dx + [\overline{\phi}^j F_{p^j}]_a^b.$$

Since u is an extremal, it is a solution of the Euler equations, so the integrand on the right is identically equal to 0. Hence, (3.16) becomes

$$0 = \left[\sum_{j=1}^N \overline{\phi}^j F_{p^j}\right]_a^b + \int_a^b \left\{\overline{\mu} F_x + \overline{\mu}' \left[F - \sum_{j=1}^N u^{j\prime} F_{p^j}\right]\right\} dx.$$

We also integrate the integral in this equation by parts, after which it becomes

$$\begin{aligned} 0 = &\left[\sum_{j=1}^N \overline{\phi}^j F_{p^j} + \overline{\mu}\left(F - \sum_{j=1}^N u^{j\prime} F_{p^j}\right)\right]_a^b \\ &+ \int_a^b \overline{\mu}\left[F_x - \frac{d}{dx}\left(F - \sum_{j=1}^N u^{j\prime} F_{p^j}\right)\right] dx. \end{aligned} \tag{3.17}$$

The first term on the right will be recognized as the predicted conserved quantity. The term in square brackets in the integrand resolves as follows:

$$\begin{aligned} F_x - \frac{d}{dx}\left(F - \sum_{j=1}^N u^{j\prime} F_{p^j}\right) &= F_x - F_x - \sum_{j=1}^N u^{j\prime} F_{u^j} - \sum_{j=1}^N u^{j\prime\prime} F_{p^j} \\ &\quad + \sum_{j=1}^N u^{j\prime\prime} F_{p^j} + \sum_{j=1}^N u^{j\prime} \frac{d}{dx} F_{p^j} \\ &= \sum_{j=1}^N u^{j\prime}\left(\frac{d}{dx} F_{p^j} - F_{u^j}\right) \\ &= 0, \end{aligned}$$

since the extremal u satisfies Euler's equations.

Hence (3.17) becomes

$$\left[\sum_{j=1}^N \overline{\phi}^j F_{p^j} + \overline{\mu}\left(F - \sum_{j=1}^N u^{j\prime} F_{p^j}\right)\right]_a^b = 0.$$

Since we have assumed that this is true for every interval $I = (a, b)$, we conclude that the quantity

$$\sum_{j=1}^N \phi^j F_{p^j} + \mu\left(F - \sum_{j=1}^N p^j F_{p^j}\right) \tag{3.18}$$

is constant along the extremal u.

Remark 3.2. It is interesting to express the conserved quantity in (3.18) in terms of the Hamiltonian variables. The generalized momentum is $\pi^j = F_{p^j}$ and the Hamiltonian is $H = \sum_{j=1}^{N} p^j F_{p^j} - F$. Hence the conserved quantity is

$$\sum_{j=1}^{N} \phi^j \pi^j - \mu H.$$

Department of Mathematics, MS 136, Rice University, PO Box 1892, Houston, TX 77251

E-mail address: polking@rice.edu