EXERCISES

Chapter 4

Section 4.1

1. If $y_1(t) = e^{3t}$, then

$$\begin{aligned} y_1'' - y_1' - 6y_1 &= (e^{3t})'' - (e^{3t})' - 6(e^{3t}) \\ &= 9e^{3t} - 3e^{3t} - 6e^{3t} \\ &= 0. \end{aligned}$$

If $y_2(t) = e^{-2t}$, then

$$\begin{aligned} y_2'' - y_2' - 6y_2 &= (e^{-2t})'' - (e^{-2t})' - 6(e^{-2t}) \\ &= 4e^{-2t} + 2e^{-2t} - 6e^{-2t} \\ &= 0. \end{aligned}$$

Finally, if $y(t) = C_1e^{3t} + C_2e^{-2t}$, then

$$\begin{aligned} &y'' - y' - 6y \\ &\quad = (C_1e^{3t} + C_2e^{-2t})'' - (C_1e^{3t} + C_2e^{-2t})' - 6(C_1e^{3t} + C_2e^{-2t}) \\ &\quad = 9C_1e^{3t} + 4C_2e^{-2t} - 3C_1e^{3t} + 2C_2e^{-2t} - 6C_1e^{3t} - 6C_2e^{-2t} \\ &\quad = 0. \end{aligned}$$

2.

3. If $y_1(t) = e^t \cos t$, then

$$\begin{aligned} y_1'(t) &= e^t(-\sin t) + e^t \cos t \\ &= e^t(\cos t - \sin t), \\ y_1''(t) &= e^t(-\sin t - \cos t) + e^t(\cos t - \sin t) \\ &= -2e^t \sin t, \end{aligned}$$

and

$$\begin{aligned} &y_1'' - 2y_1' + 2y_1 \\ &\quad = -2e^t \sin t - 2e^t(\cos t - \sin t) + 2e^t \cos t \\ &\quad = 0. \end{aligned}$$

If $y_2(t) = e^t \sin t$, then

$$\begin{aligned} y_2'(t) &= e^t \cos t + e^t \sin t \\ &= e^t(\cos t + \sin t), \\ y_2''(t) &= e^t(-\sin t + \cos t) + e^t(\cos t + \sin t) \\ &= 2e^t \cos t, \end{aligned}$$

and

$$\begin{aligned} &y_2'' - 2y_2' + 2y_2 \\ &\quad = 2e^t \cos t - 2e^t(\cos t + \sin t) + 2e^t \sin t \\ &\quad = 0. \end{aligned}$$

Finally, if $y(t) = C_1 e^t \cos t + C_2 e^t \sin t$, or if $y(t) = e^t(C_1 \cos t + C_2 \sin t)$, then

$$\begin{aligned} y'(t) &= e^t(-C_1 \sin t + C_2 \cos t) + e^t(C_1 \cos t + C_2 \sin t) \\ &= e^t((C_1 + C_2)\cos t + (-C_1 + C_2)\sin t), \\ y''(t) &= e^t((C_1 + C_2)(-\sin t) + (-C_1 + C_2)\cos t) \\ &\quad + e^t((C_1 + C_2)\cos t + (-C_1 + C_2)\sin t) \\ &= 2e^t(C_2 \cos t - C_1 \sin t), \end{aligned}$$

and

$$\begin{aligned} & y'' - 2y' + 2y \\ &= 2e^t(C_2 \cos t - C_1 \sin t) \\ &\quad - 2e^t((C_1 + C_2)\cos t + (-C_1 + C_2)\sin t) \\ &\quad\quad + 2e^t(C_1 \cos t + C_2 \sin t) \\ &= 0. \end{aligned}$$

4.

5. We leave it to our readers to first check that $y_1(t) = e^{-t}$ and $y_2(t) = e^{2t}$ are solutions of $y'' - y' - 2y = 0$. Next note that

$$\frac{y_1(t)}{y_2(t)} = \frac{e^{-t}}{e^{2t}} = e^{-3t} \neq c,$$

where c is some constant. Therefore $y_1(t)$ is not a constant multiple of $y_2(t)$ and the solutions are linearly independent. Further,

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2e^{2t} \end{vmatrix} = 3e^t,$$

which is never equal to zero. Therefore, the solutions $y_1(t)$ and $y_2(t)$ are linearly independent and form a fundamental set of solutions.

6.

7. We leave it to our readers to first check that $y_1(t) = e^{-2t} \cos 3t$ and $y_2(t) = e^{-2t} \sin 3t$ are solutions of $y'' + 4y' + 13y = 0$. Next note that

$$\frac{y_1(t)}{y_2(t)} = \frac{e^{-2t} \cos 3t}{e^{-2t} \sin 3t} = \cot 3t \neq c,$$

where c is some constant. Therefore $y_1(t)$ is not a constant multiple of $y_2(t)$ and the solutions are linearly independent. Further,

$$\begin{aligned} W(t) &= \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \\ &= \begin{vmatrix} e^{-2t} \cos 3t & e^{-2t} \sin 3t \\ e^{-2t}(-3 \sin 3t - 2 \cos 3t) & e^{-2t}(3 \cos 3t - 2 \sin 3t) \end{vmatrix} \\ &= e^{-4t}(3 \cos^2 3t - 2 \cos 3t \sin 3t) - e^{-4t}(-3 \sin^2 3t - 2 \cos 3t \sin 3t) \\ &= -3e^{-4t}, \end{aligned}$$

which is never equal to zero. Therefore, the solutions $y_1(t)$ and $y_2(t)$ are linearly independent and form a fundamental set of solutions.

8.

9. If $y_1(t) = t^2$ and $y_2(t) = t|t|$, then

$$\frac{y_1(t)}{y_2(t)} = \frac{t^2}{t|t|} = \begin{cases} -1, & t < 0, \\ 1, & t > 0. \end{cases}$$

Thus, $y_1(t)$ is not a constant multiple of $y_2(t)$ on $(-\infty, +\infty)$. However, the Wronskian

$$\begin{aligned} W(t) &= \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \\ &= \begin{vmatrix} t^2 & t|t| \\ 2t & 2|t| \end{vmatrix} \\ &= 2t^2|t| - 2t^2|t| \\ &= 0. \end{aligned}$$

Thus, the Wronskian is identically zero on $(-\infty, +\infty)$, seemingly contradicting Proposition 1.26, that is, until one realizes that the hypothesis that requires that y_1 and y_2 are solutions of the differential equation $y'' + p(t)y' + q(t)y = 0$ is unsatisfied.

10.

11. If $y_1(t) = \cos 4t$, then

$$y_1'' + 16y_1 = -16\cos 4t + 16\cos 4t = 0,$$

and if $y_2(t) = \sin 4t$, then

$$y_2'' + 16y_2 = -16\sin 4t + 16\sin 4t = 0.$$

Furthermore,

$$\frac{y_1(t)}{y_2(t)} = \frac{\cos 4t}{\sin 4t} = \cot 4t,$$

which is nonconstant. Thus, y_1 is not a constant multiple of y_2 and the solutions $y_1(t) = \cos 4t$ and $y_2(t) = \sin 4t$ form a fundamental set of solutions. Thus, the general solution of $y'' + 16y = 0$ is

$$y(t) = C_1 \cos 4t + C_2 \sin 4t,$$

and its derivative is

$$y'(t) = -4C_1 \sin 4t + 4C_2 \cos 4t.$$

The initial conditions, $y(0) = 2$ and $y'(0) = -1$ lead to the equations

$$\begin{aligned} 2 &= C_1 \\ -1 &= 4C_2 \end{aligned}$$

and the constants $C_1 = 2$ and $C_2 = -1/4$. Thus, the solution of the initial value problem is

$$y(t) = 2\cos 4t - \frac{1}{4}\sin 4t.$$

12.

13. If $y_1(t) = e^{-4t}$, then

$$y_1'' + 8y_1' + 16y_1 = 16e^{-4t} + 8(-4e^{-4t}) + 16e^{-4t} = 0.$$

Thus, y_1 is a solution. If $y_2(t) = te^{-4t}$, then

$$\begin{aligned} y_2'(t) &= e^{-4t}(1 - 4t), \text{ and} \\ y_2''(t) &= e^{-4t}(-8 + 16t). \end{aligned}$$

Thus,

$$\begin{aligned} &y_2'' + 8y_2' + 16y_2 \\ &\quad = e^{-4t}(-8 + 16t) + 8e^{-4t}(1 - 4t) + 16te^{-4t} \\ &\quad = 0. \end{aligned}$$

Thus, y_2 is a solution. Because

$$\frac{y_1(t)}{y_2(t)} = \frac{e^{-4t}}{te^{-4t}} = \frac{1}{t}$$

is nonconstant, y_1 and y_2 are independent and form a fundamental set of solutions. Thus, the general solution is

$$y(t) = C_1e^{-4t} + C_2te^{-4t}.$$

Substituting the initial condition $y(0) = 2$ provides $C_1 = 2$. After a bit of work, the derivative of the general solution is

$$y'(t) = e^{-4t}((C_2 - 4C_1) - 4C_2t).$$

Substituting the initial condition $y'(0) = -1$ gives

$$-1 = C_2 - 4C_1,$$

and the fact that $C_1 = 2$ provides us with

$$\begin{aligned} -1 &= C_2 - 8 \\ C_2 &= 7. \end{aligned}$$

Thus, the solution is

$$y(t) = 2e^{-4t} + 7te^{-4t}.$$

14. (a)
(b)

15. If $y_1(t) = t$, then

$$t^2y_1'' - 2ty_1' + 2y_1 = t^2(0) - 2t(1) + 2t = 0.$$

Thus, $y_1(t) = t$ is a solution. Let $y_2(t) = v(t)y_1(t) = tv(t)$, where v is to be determined. Then,

$$\begin{aligned} y_2 &= tv \\ y_2' &= tv' + v \\ y_2'' &= tv'' + 2v'. \end{aligned}$$

Substituting these into the differential equation,

$$\begin{aligned}0 &= t^2y_2'' - 2ty_2' + 2y_2\\ &= t^2(tv'' + 2v') - 2t(tv' + v) + 2tv\\ &= t^3v''.\end{aligned}$$

Because t^3 is not identically equal to zero, $v'' = 0$ and

$$\begin{aligned}v' &= k_1\\ v &= k_1t + k_2.\end{aligned}$$

Choose $k_1 = 1$ and $k_2 = 0$, which gives $v(t) = t$ and $y_2(t) = v(t)y_1(t) = tt = t^2$. Checking, we see that

$$t^2y_2'' - 2ty_2' + 2y_2 = t^2(2) - 2t(2t) + 2(t^2) = 0,$$

making $y_2(t) = t^2$ a solution. Note that

$$\frac{y_1(t)}{y_2(t)} = \frac{t}{t^2} = \frac{1}{t}$$

is nonconstant, so y_1 and y_2 are independent and form a fundamental set of solutions. Thus, the general solution is

$$y(t) = C_1t + C_2t^2.$$

16.

17. If $y_1(t) = t$, then

$$t^2y_1'' - 3ty_1' + 3y_1 = t^2(0) - 3t(1) + 3t = 0,$$

making $y_1(t) = t$ a solution. Let $y_2(t) = v(t)y_1(t) = tv(t)$, with v to be determined. Thus,

$$\begin{aligned}y_2 &= tv\\ y_2' &= tv' + v\\ y_2'' &= 2v' + tv''.\end{aligned}$$

Substituting these into the differential equation,

$$\begin{aligned}0 &= t^2y_2'' - 3ty_2' + 3y_2\\ &= t^2(2v' + tv'') - 3t(tv' + v) + 3tv\\ &= t^2(tv'' - v').\end{aligned}$$

Because t^2 is not identically zero,

$$\begin{aligned}tv'' - v' &= 0\\ \frac{v''}{v'} &= \frac{1}{t}\\ \ln v' &= \ln t\\ v' &= t\\ v &= \frac{1}{2}t^2.\end{aligned}$$

Thus,

$$y_2(t) = v(t)y_1(t) = \frac{1}{2}t^2(t) = \frac{1}{2}t^3.$$

Readers should check that this is a solution of the differential equation. Further,

$$\frac{y_1(t)}{y_2(t)} = \frac{t}{(1/2)t^3} = \frac{2}{t^2}$$

is nonconstant, so y_1 and y_2 are independent and form a fundamental set of solutions. Thus, the general solution is

$$y(t) = C_1 t + C_2 t^3,$$

where we have absorbed the $1/2$ into the arbitrary constant C_2.

18.

Section 4.2

1. First, solve the differential equation $y'' + 2y' - 3y = 0$ for the highest derivative of y present in the equation.

$$y'' = -2y' + 3y$$

Next, set $v = y'$. Then

$$v' = y'' = -2y' + 3y = -2v + 3y.$$

Thus, we now have the following system of first order equations.

$$\begin{aligned} y' &= v \\ v' &= -2v + 3y \end{aligned}$$

2.

3. First, solve the differential equation $y'' + 3y' + 4y = 2\cos 2t$ for the highest derivative of y present in the equation.

$$y'' = -3y' - 4y + 2\cos 2t$$

Next, set $v = y'$. Then

$$v' = y'' = -3y' - 4y + 2\cos 2t = -3v - 4y + 2\cos 2t.$$

Thus, we now have the following system of first order equations.

$$\begin{aligned} y' &= v \\ v' &= -3v - 4y + 2\cos 2t \end{aligned}$$

4.

5. First, solve the differential equation $y'' + \mu(t^2 - 1)y' + y = 0$ for the highest derivative of y present in the equation.

$$y'' = -\mu(t^2 - 1)y' - y$$

Next, set $v = y'$. Then

$$v' = y'' = -\mu(t^2 - 1)y' - y = -\mu(t^2 - 1)v - y.$$

Thus, we now have the following system of first order equations.

$$\begin{aligned} y' &= v \\ v' &= -\mu(t^2 - 1)v - y \end{aligned}$$

6.

7. First, solve the differential equation $LQ'' + RQ' + (1/C)Q = E(t)$ for the highest derivative of Q present in the equation.

$$Q'' = -\frac{R}{L}Q' - \frac{1}{LC}Q + \frac{1}{L}E(t)$$

Next, set $I = Q'$. Then

$$I' = Q'' = -\frac{R}{L}Q' - \frac{1}{LC}Q + \frac{1}{L}E(t) = -\frac{R}{L}I - \frac{1}{LC}Q + \frac{1}{L}E(t).$$

Thus, we now have the following system of first order equations.

$$\begin{aligned} Q' &= I \\ I' &= -\frac{R}{L}I - \frac{1}{LC}Q + \frac{1}{L}E(t) \end{aligned}$$

8.

9. Our solver requires that we first change the second order equation $my'' + \mu y' + ky = 0$ into a system of two first order equations. If we let $y' = v$, then

$$v' = y'' = -\frac{\mu}{m}y' - \frac{k}{m}y = -\frac{\mu}{m}v - \frac{k}{m}y,$$

which leads to the following system of first order equations.

$$\begin{aligned} y' &= v \\ v' &= -\frac{\mu}{m}v - \frac{k}{m}y \end{aligned}$$

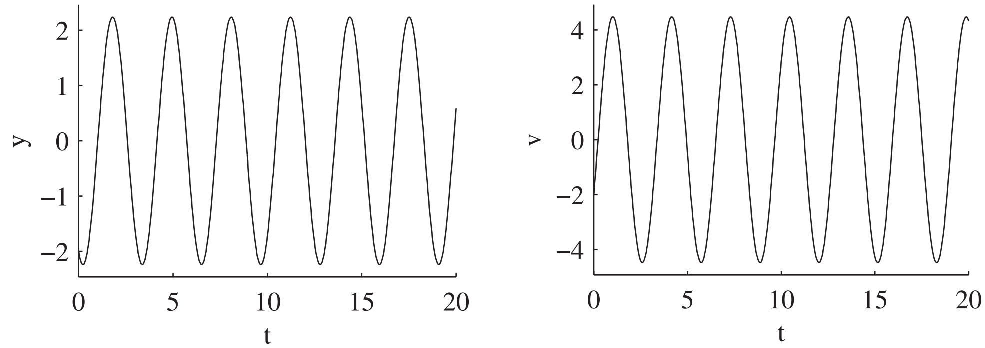
Now, if $\mu = 0\,\text{kg/s}$, $k = 4\,\text{kg/s}^2$, $y(0) = -2\,\text{m}$, and $y'(0) = -2\,\text{m/s}$, then

$$\begin{aligned} y' &= v \\ v' &= -4y, \end{aligned}$$

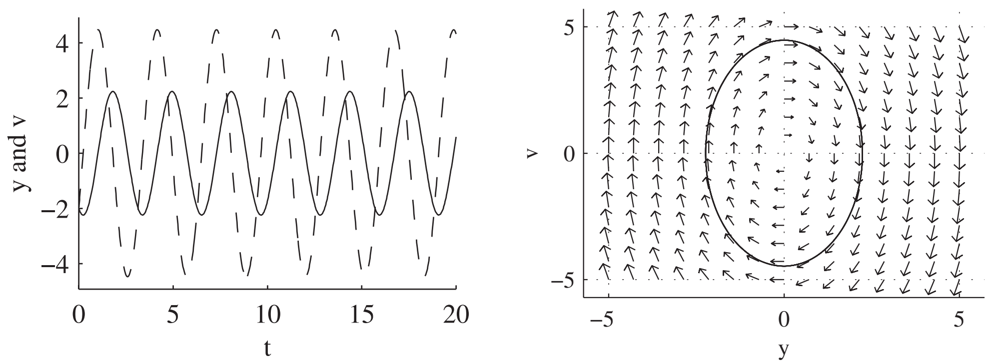
and

$$\begin{aligned} y(0) &= -2 \\ v(0) &= y'(0) = -2. \end{aligned}$$

Entering this system in our solver, we generate plots of position versus time and velocity versus time.



Our solver also generates a combined plot of both position and velocity versus time, and a plot of velocity versus position in the phase plane.



10.

11. Our solver requires that we first change the second order equation $my'' + \mu y' + ky = 0$ into a system of two first order equations. If we let $y' = v$, then

$$v' = y'' = -\frac{\mu}{m}y' - \frac{k}{m}y = -\frac{\mu}{m}v - \frac{k}{m}y,$$

which leads to the following system of first order equations.

$$\begin{aligned} y' &= v \\ v' &= -\frac{\mu}{m}v - \frac{k}{m}y \end{aligned}$$

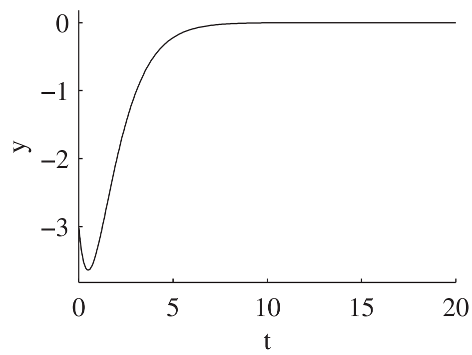
Now, if $m = 1$ kg, $\mu = 2$ kg/s, $k = 1$ kg/s^2, $y(0) = -3$ m, and $y'(0) = -2$ m/s, then

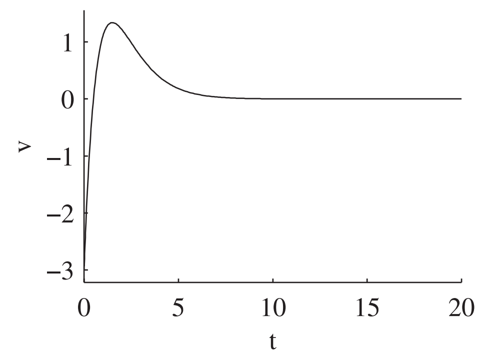
$$\begin{aligned} y' &= v \\ v' &= -2v - y, \end{aligned}$$

and

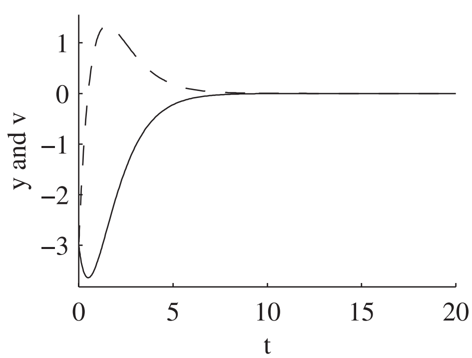
$$\begin{aligned} y(0) &= -3 \\ v(0) &= y'(0) = -2. \end{aligned}$$

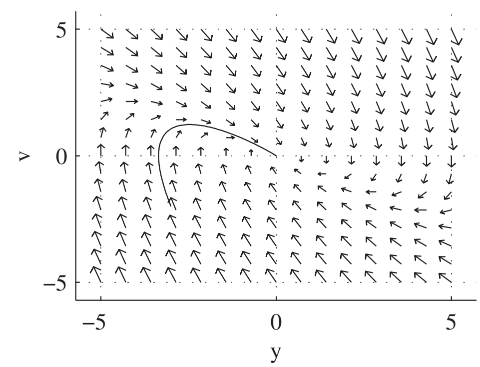
Entering this system in our solver, we generate plots of position versus time and velocity versus time.





Our solver also generates a combined plot of both position and velocity versus time, and a plot of velocity versus position in the phase plane.





12.

13. Our solver requires that we first change the second order equation $my'' + \mu y' + ky = 0$ into a system of two first order equations. If we let $y' = v$, then

$$v' = y'' = -\frac{\mu}{m}y' - \frac{k}{m}y = -\frac{\mu}{m}v - \frac{k}{m}y,$$

which leads to the following system of first order equations.

$$\begin{aligned} y' &= v \\ v' &= -\frac{\mu}{m}v - \frac{k}{m}y \end{aligned}$$

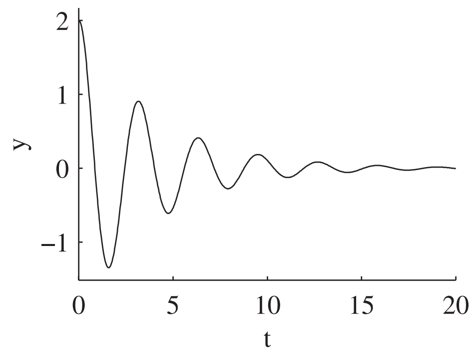
Now, if $m = 1\,\text{kg}$, $\mu = 0.5\,\text{kg/s}$, $k = 4\,\text{kg/s}^2$, $y(0) = 2\,\text{m}$, and $y'(0) = 0\,\text{m/s}$, then

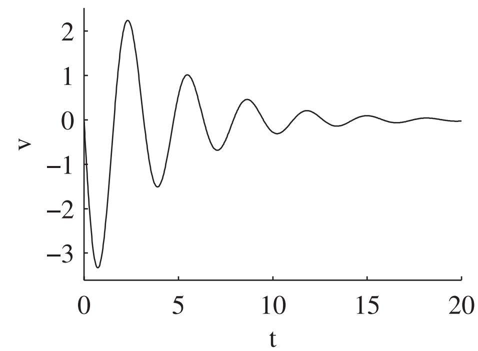
$$\begin{aligned} y' &= v \\ v' &= -0.5v - 4y, \end{aligned}$$

and

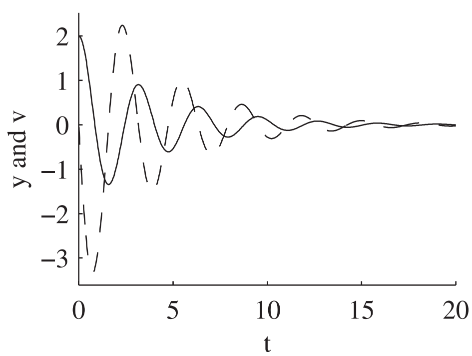
$$y(0) = 2$$
$$v(0) = y'(0) = 0.$$

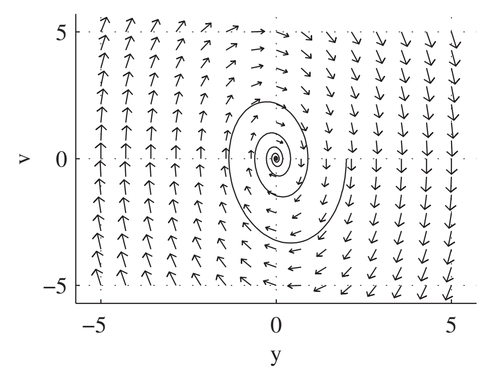
Entering this system in our solver, we generate plots of position versus time and velocity versus time.





Our solver also generates a combined plot of both position and velocity versus time, and a plot of velocity versus position in the phase plane.





14.

15. Our solver requires that we first change the second order equation $my'' + \mu y' + ky = 0$ into a system of two first order equations. If we let $y' = v$, then

$$v' = y'' = -\frac{\mu}{m}y' - \frac{k}{m}y = -\frac{\mu}{m}v - \frac{k}{m}y,$$

which leads to the following system of first order equations.

$$y' = v$$
$$v' = -\frac{\mu}{m}v - \frac{k}{m}y$$

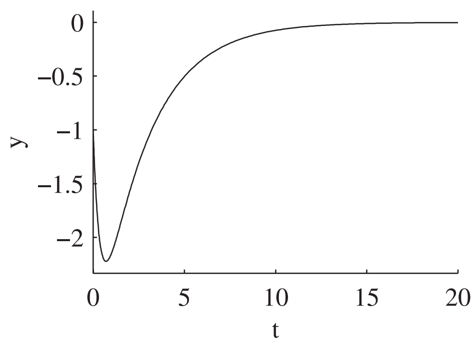
Now, if $m = 1\,\text{kg}$, $\mu = 2\,\text{kg/s}$, $k = 1\,\text{kg/s}^2$, $y(0) = 1\,\text{m}$, and $y'(0) = -5\,\text{m/s}$, then

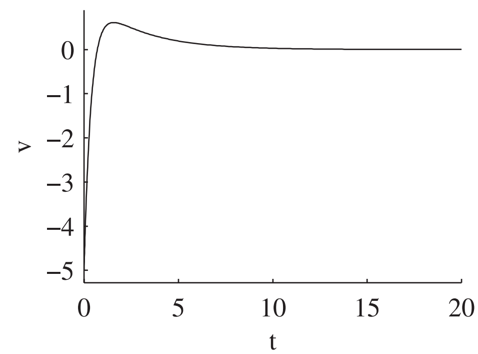
$$\begin{aligned} y' &= v \\ v' &= -3v - y, \end{aligned}$$

and

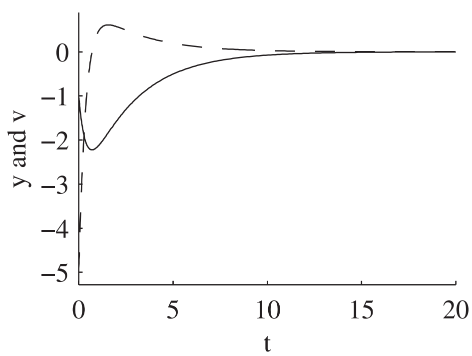
$$\begin{aligned} y(0) &= -1 \\ v(0) &= y'(0) = -5. \end{aligned}$$

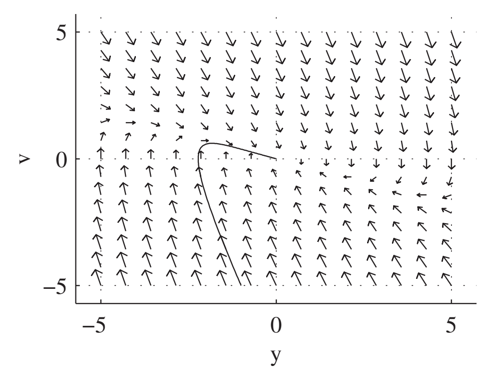
Entering this system in our solver, we generate plots of position versus time and velocity versus time.





Our solver also generates a combined plot of both position and velocity versus time, and a plot of velocity versus position in the phase plane.





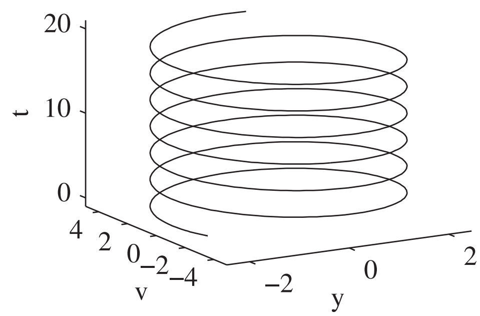
16.

17. (a) In Figure 3, the peaks of the curve $t \to y(t)$ occur where $y' = v = 0$, and these are the points where the curve $t \to v(t)$ crosses the t-axis.

(b) In Figure 3, the peaks of the curve $t \to v(t)$ occur slightly before the curve $t \to y(t)$ crosses the t-axis. This is because the velocity has maximum magnitude where $v' = a = y'' = 0$. Since the damping constant $\mu > 0$, at these points the differential equation $my'' + \mu y' + ky = 0$ gives $y = -\mu v/k$. Thus $y \neq 0$ and has a sign opposite to that of v.

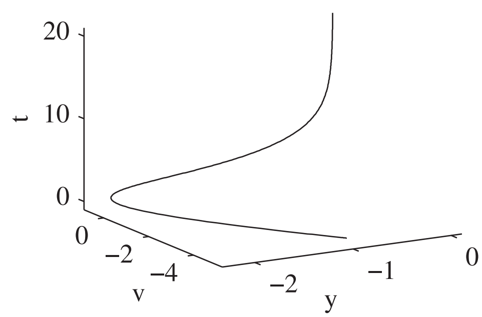
18. (a)
(b)
(c)

19. A plot of $t \to (y(t), v(t), t)$ for the system in Exercise 9.



20.

21. A plot of $t \to (y(t), v(t), t)$ for the system in Exercise 15.



22.

23. Our solver requires that we first change the second order equation $LQ'' + RQ' + (1/C)Q = 2\cos 2t$ into a system of two first order equations. If we let $Q' = I$, then

$$I' = Q'' = -\frac{R}{L}Q' - \frac{1}{LC}Q + 2\cos 2t = -\frac{R}{L}I - \frac{1}{LC}Q + 2\cos 2t,$$

which leads to the following system of first order equations.

$$\begin{aligned} Q' &= I \\ I' &= -\frac{R}{L}I - \frac{1}{LC}Q + 2\cos 2t \end{aligned}$$

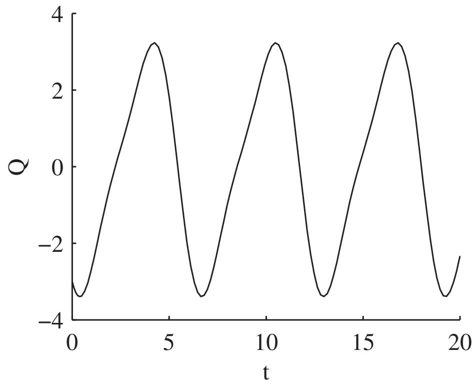
Now, if $L = 1\,\text{H}$, $R = 0\,\Omega$, $C = 1\,\text{F}$, $Q(0) = -3\,\text{C}$, and $I(0) = -2\,\text{A}$, then

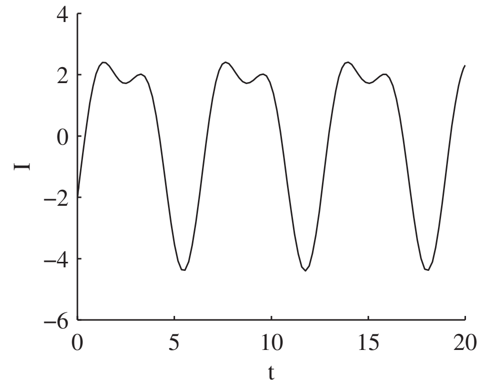
$$\begin{aligned} Q' &= I \\ I' &= -Q + 2\cos 2t, \end{aligned}$$

and

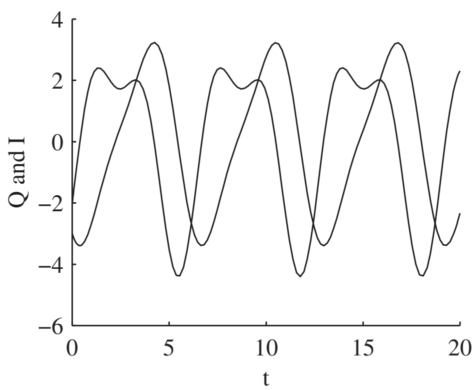
$$Q(0) = -3$$
$$I(0) = -2.$$

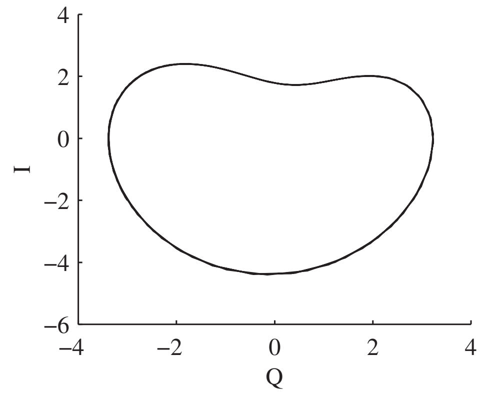
Entering this system in our solver, we generate plots of charge versus time and current versus time.





Our solver also generates a combined plot of both charge and current versus time, and a plot of current versus charge in the phase plane.





24.

25. Our solver requires that we first change the second order equation $LQ'' + RQ' + (1/C)Q = 2\cos 2t$ into a system of two first order equations. If we let $Q' = I$, then

$$I' = Q'' = -\frac{R}{L}Q' - \frac{1}{LC}Q + 2\cos 2t = -\frac{R}{L}I - \frac{1}{LC}Q + 2\cos 2t,$$

which leads to the following system of first order equations.

$$Q' = I$$
$$I' = -\frac{R}{L}I - \frac{1}{LC}Q + 2\cos 2t$$

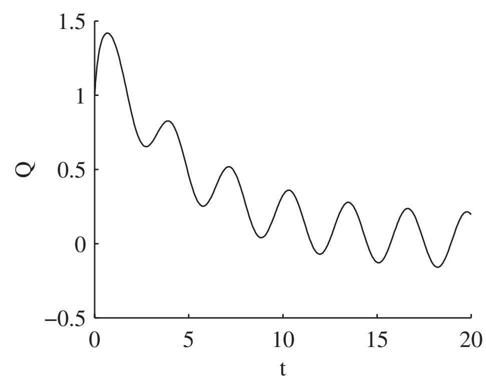
Now, if $L = 1\,\mathrm{H}$, $R = 5\,\Omega$, $C = 1\,\mathrm{F}$, $Q(0) = 1\,\mathrm{C}$, and $I(0) = 2\,\mathrm{A}$, then

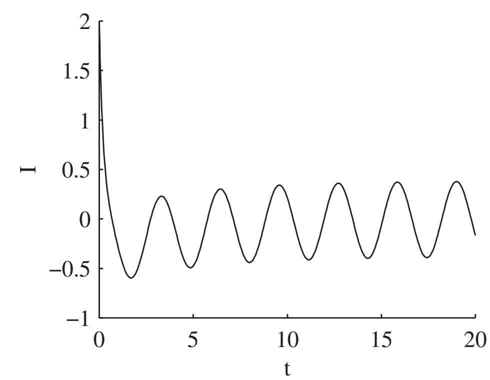
$$\begin{aligned} Q' &= I \\ I' &= -5I - Q + 2\cos 2t, \end{aligned}$$

and

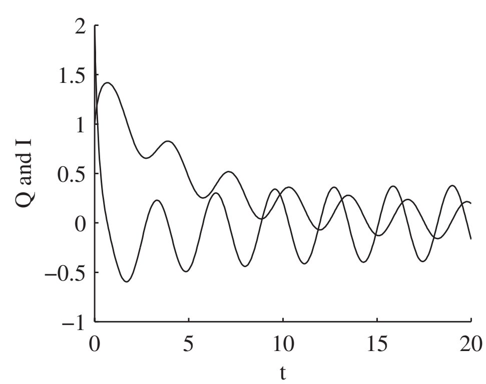
$$\begin{aligned} Q(0) &= 1 \\ I(0) &= 2. \end{aligned}$$

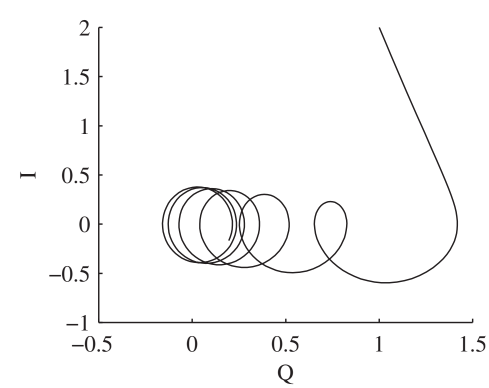
Entering this system in our solver, we generate plots of charge versus time and current versus time.





Our solver also generates a combined plot of both charge and current versus time, and a plot of current versus charge in the phase plane.





26.

27. Our solver requires that we first change the second order equation $LQ'' + RQ' + (1/C)Q = 2\cos 2t$ into a system of two first order equations. If we let $Q' = I$, then

$$I' = Q'' = -\frac{R}{L}Q' - \frac{1}{LC}Q + 2\cos 2t = -\frac{R}{L}I - \frac{1}{LC}Q + 2\cos 2t,$$

which leads to the following system of first order equations.

$$Q' = I$$
$$I' = -\frac{R}{L}I - \frac{1}{LC}Q + 2\cos 2t$$

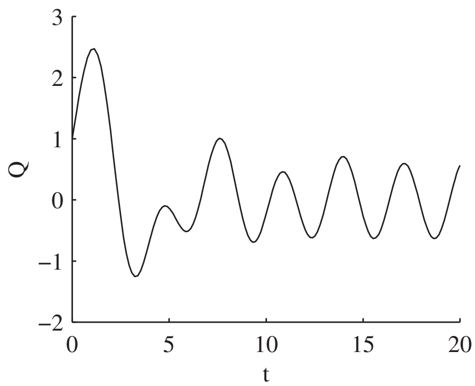
Now, if $L = 1\,\text{H}$, $R = 0.5\,\Omega$, $C = 1\,\text{F}$, $Q(0) = 1\,\text{C}$, and $I(0) = 2\,\text{A}$, then

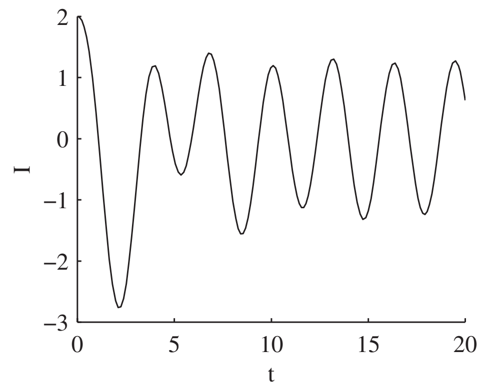
$$Q' = I$$
$$I' = -0.5I - Q + 2\cos 2t,$$

and

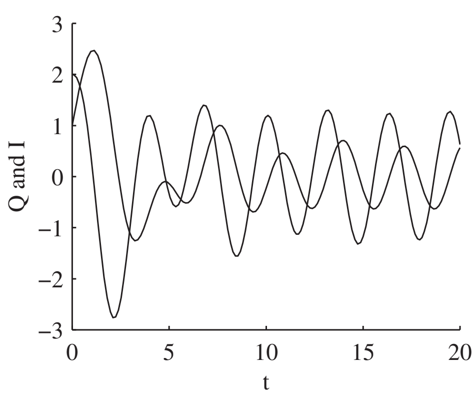
$$Q(0) = 1$$
$$I(0) = 2.$$

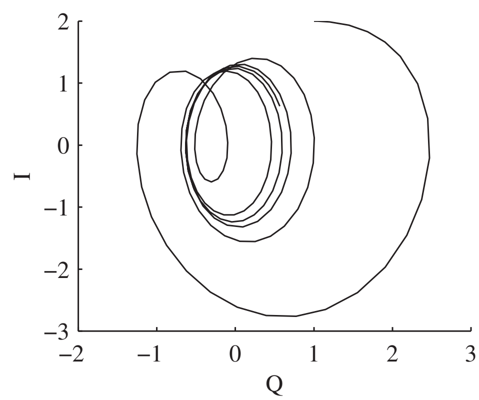
Entering this system in our solver, we generate plots of charge versus time and current versus time.





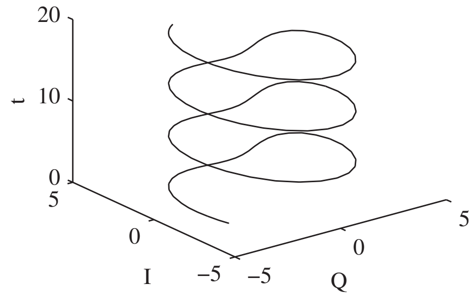
Our solver also generates a combined plot of both charge and current versus time, and a plot of current versus charge in the phase plane.





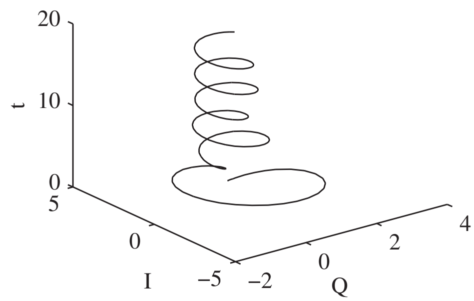
28.

29. A plot of $t \to (Q(t), I(t), t)$ for the circuit of Exercise 23.



30.

31. A plot of $t \to (Q(t), I(t), t)$ for the circuit of Exercise 27.



32.

Section 4.3

1. Let $y = e^{\lambda t}$ in $y'' - y' - 2y = 0$ to obtain

$$\begin{aligned}\lambda^2 e^{\lambda t} - \lambda e^{\lambda t} - 2e^{\lambda t} &= 0\\ e^{\lambda t}(\lambda^2 - \lambda - 2) &= 0.\end{aligned}$$

Because $e^{\lambda t} \neq 0$, we arrive at the characteristic equation

$$\begin{aligned}\lambda^2 - \lambda - 2 &= 0\\ (\lambda - 2)(\lambda + 1) &= 0,\end{aligned}$$

and roots $\lambda = 2$ and $\lambda = -1$. Because the roots are distinct, the solutions $y_1(t) = e^{2t}$ and $y_2(t) = e^{-t}$ form a fundamental set of solutions and the general solution is

$$y(t) = C_1 e^{2t} + C_2 e^{-t}.$$

2.

3. Let $y = e^{\lambda t}$ in $y'' + y' - 12y = 0$ to obtain

$$\begin{aligned}\lambda^2 e^{\lambda t} + \lambda e^{\lambda t} - 12e^{\lambda t} &= 0\\ e^{\lambda t}(\lambda^2 + \lambda - 12) &= 0.\end{aligned}$$

Because $e^{\lambda t} \neq 0$, we arrive at the characteristic equation

$$\lambda^2 + \lambda - 12 = 0$$
$$(\lambda + 4)(\lambda - 3) = 0,$$

and roots $\lambda = -4$ and $\lambda = 3$. Because the roots are distinct, the solutions $y_1(t) = e^{-4t}$ and $y_2(t) = e^{3t}$ form a fundamental set of solutions and the general solution is

$$y(t) = C_1 e^{-4t} + C_2 e^{3t}.$$

4.

5. Let $y = e^{\lambda t}$ in $6y'' + y' - y = 0$ to obtain

$$6\lambda^2 e^{\lambda t} + \lambda e^{\lambda t} - e^{\lambda t} = 0$$
$$e^{\lambda t}(6\lambda^2 + \lambda - 1) = 0.$$

Because $e^{\lambda t} \neq 0$, we arrive at the characteristic equation

$$6\lambda^2 + \lambda - 1 = 0$$
$$(3\lambda - 1)(2\lambda + 1) = 0,$$

and roots $\lambda = 1/3$ and $\lambda = -1/2$. Because the roots are distinct, the solutions $y_1(t) = e^{(1/3)t}$ and $y_2(t) = e^{-(1/2)t}$ form a fundamental set of solutions and the general solution is

$$y(t) = C_1 e^{t/3} + C_2 e^{-t/2}.$$

6.

7. If $y'' + y = 0$, then the characteristic equation is

$$\lambda^2 + 1 = 0.$$

The roots of the characteristic equation are $\pm i$, leading to the complex solutions

$$z(t) = e^{it} \qquad \text{and} \qquad \overline{z}(t) = e^{-it}.$$

However, by Euler's identity,

$$z(t) = \cos t + i \sin t,$$

and the real and imaginary parts of z lead to a fundamental set of real solutions, $y_1(t) = \cos t$ and $y_2(t) = \sin t$. Hence, the general solution is

$$y(t) = C_1 \cos t + C_2 \sin t.$$

8.

9. If $y'' + 4y' + 5y = 0$, then the characteristic equation is

$$\lambda^2 + 4\lambda + 5 = 0.$$

The roots of the characteristic equation are

$$z = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm 2i,$$

leading to the complex solutions

$$z(t) = e^{(-2+2i)t} \qquad \text{and} \qquad \overline{z}(t) = e^{(-2-2i)t}.$$

However, by Euler's identity,

$$z(t) = e^{-2t}e^{i2t} = e^{-2t}(\cos 2t + i \sin 2t),$$

and the real and imaginary parts of z lead to a fundamental set of real solutions, $y_1(t) = e^{-2t}\cos 2t$ and $y_2(t) = e^{-2t}\sin 2t$. Hence, the general solution is

$$y(t) = C_1 e^{-2t}\cos 2t + C_2 e^{-2t}\sin 2t = e^{-2t}(C_1\cos 2t + C_2 \sin 2t).$$

10.

11. If $y'' + 2y = 0$, then the characteristic equation is

$$\lambda^2 + 2 = 0.$$

The roots of the characteristic equation are $\pm\sqrt{2}i$, leading to the complex solutions

$$z(t) = e^{\sqrt{2}it} \qquad \text{and} \qquad \overline{z}(t) = e^{-\sqrt{2}it}.$$

However, by Euler's identity,

$$z(t) = \cos\sqrt{2}t + i\sin\sqrt{2}t,$$

and the real and imaginary parts of z lead to a fundamental set of real solutions, $y_1(t) = \cos\sqrt{2}t$ and $y_2(t) = \sin\sqrt{2}t$. Hence, the general solution is

$$y(t) = C_1\cos\sqrt{2}t + C_2\sin\sqrt{2}t.$$

12.

13. If $y'' - 4y' + 4y = 0$, then the characteristic equation is

$$\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0.$$

Hence, the characteristic equation has a single, double root, $\lambda = 2$. Therefore, $y_1(t) = e^{2t}$ and $y_2(t) = te^{2t}$ form a fundamental set of real solutions. Hence, the general solution is

$$y(t) = C_1e^{2t} + C_2te^{2t} = (C_1 + C_2t)e^{2t}.$$

14.

15. If $4y'' + 4y' + y = 0$, then the characteristic equation is

$$4\lambda^2 + 4\lambda + 1 = (2\lambda + 1)^2 = 0.$$

Hence, the characteristic equation has a single, double root, $\lambda = -1/2$. Therefore, $y_1(t) = e^{-(1/2)t}$ and $y_2(t) = te^{-(1/2)t}$ form a fundamental set of real solutions. Hence, the general solution is

$$y(t) = C_1e^{-t/2} + C_2te^{-t/2} = (C_1 + C_2t)e^{-t/2}.$$

16.

17. If $16y'' + 8y' + y = 0$, then the characteristic equation is

$$16\lambda^2 + 8\lambda + 1 = (4\lambda + 1)^2 = 0.$$

Hence, the characteristic equation has a single, double root, $\lambda = -1/4$. Therefore, $y_1(t) = e^{-(1/4)t}$ and $y_2(t) = te^{-(1/4)t}$ form a fundamental set of real solutions. Hence, the general solution is

$$y(t) = C_1 e^{-t/4} + C_2 t e^{-t/4} = (C_1 + C_2 t)e^{-t/4}.$$

18.

19. If $y'' - y' - 2y = 0$, then the characteristic equation is

$$\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0,$$

with roots $\lambda = 2$ and $\lambda = -1$. This leads to the general solution

$$y(t) = C_1 e^{2t} + C_2 e^{-t}.$$

Using the initial condition $y(0) = -1$ provides

$$-1 = C_1 + C_2.$$

Differentiating the general solution,

$$y'(t) = 2C_1 e^{2t} - C_2 e^{-t},$$

then using the initial condition $y'(0) = 2$ leads to

$$2 = 2C_1 - C_2.$$

These equations yield $C_1 = 1/3$ and $C_2 = -4/3$, giving the particular solution

$$y(t) = \frac{1}{3}e^{2t} - \frac{4}{3}e^{-t}.$$

20.

21. If $y'' + 25y = 0$, then the characteristic equation is $\lambda^2 + 25 = 0$, which has roots $\lambda = \pm 5i$. This leads to a pair of complex solutions

$$z(t) = e^{5it} \qquad \text{and} \qquad \overline{z}(t) = e^{-5it}.$$

By Euler's identity,

$$z(t) = \cos 5t + i \sin 5t,$$

leading to a fundamental set of real solutions $y_1(t) = \cos 5t$ and $y_2(t) = \sin 5t$ and the general solution

$$y(t) = C_1 \cos 5t + C_2 \sin 5t.$$

Next, $y(0) = 1$ provides $C_1 = 1$. Differentiating the general solution,

$$y'(t) = -5C_1 \sin 5t + 5C_2 \cos 5t,$$

and using the initial condition $y'(0) = -1$ provides $-1 = 5C_2$, or $C_2 = -1/5$. Thus, the particular solution is

$$y(t) = \cos 5t - \frac{1}{5}\sin 5t.$$

22.

23. If $y'' - 2y' - 3y = 0$, then the characteristic equation is

$$\lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0,$$

with roots $\lambda = 3$ and $\lambda = -1$. This leads to the general solution

$$y(t) = C_1 e^{3t} + C_2 e^{-t}.$$

Using the initial condition $y(0) = 2$ provides

$$2 = C_1 + C_2.$$

Differentiating the general solution,

$$y'(t) = 3C_1 e^{3t} - C_2 e^{-t},$$

then using the initial condition $y'(0) = -3$ leads to

$$-3 = 3C_1 - C_2.$$

These equations yield $C_1 = -1/4$ and $C_2 = 9/4$, giving the particular solution

$$y(t) = -\frac{1}{4}e^{3t} + \frac{9}{4}e^{-t}.$$

24.

25. If $8y'' + 2y' - y = 0$, then the characteristic equation is

$$8\lambda^2 + 2\lambda - 1 = (4\lambda - 1)(2\lambda + 1) = 0,$$

with roots $\lambda = 1/4$ and $\lambda = -1/2$. This leads to the general solution

$$y(t) = C_1 e^{t/4} + C_2 e^{-t/2}.$$

Using the initial condition $y(-1) = 1$ provides

$$1 = C_1 e^{-1/4} + C_2 e^{1/2}. \qquad (*)$$

Differentiating the general solution,

$$y'(t) = \frac{1}{4}C_1 e^{t/4} - \frac{1}{2}C_2 e^{-t/2},$$

then using the initial condition $y'(-1) = 0$ leads to

$$0 = \frac{1}{4}C_1 e^{-1/4} - \frac{1}{2}C_2 e^{1/2}. \qquad (**)$$

Subtracting 4 times equation $(**)$ from equation $(*)$, provides $1 = 3C_2 e^{1/2}$, so $C_2 = (1/3)e^{-1/2}$. Substituting this result in equation $(**)$ yields $C_1 = (2/3)e^{1/4}$. Thus, the particular solution is

$$y(t) = \frac{2}{3}e^{1/4}e^{t/4} + \frac{1}{3}e^{-1/2}e^{-t/2}$$
$$y(t) = \frac{2}{3}e^{(t+1)/4} + \frac{1}{3}e^{-(t+1)/2}.$$

26.

27. If $y'' + 12y' + 36y = 0$, then the characteristic equation is

$$\lambda^2 + 12\lambda + 36 = (\lambda + 6)^2 = 0,$$

with double root $\lambda = -6$. This leads to a fundamental set of solutions, $y_1(t) = e^{-6t}$ and $y_2(t) = te^{-6t}$, and the general solution

$$y(t) = (C_1 + C_2 t)e^{-6t}.$$

Using the initial condition $y(1) = 0$ provides

$$0 = (C_1 + C_2)e^{-6}.$$

Thus, $C_1 = -C_2$. Differentiating the general solution,

$$\begin{aligned} y'(t) &= (C_1 + C_2 t)(-6e^{-6t}) + C_2 e^{-6t} \\ &= e^{-6t}((C_2 - 6C_1) - 6C_2 t), \end{aligned}$$

then using the initial condition $y'(1) = -1$ leads to

$$\begin{aligned} -1 &= e^{-6}((C_2 - 6C_1) - 6C_2) \\ -1 &= e^{-6}(-6C_1 - 5C_2). \end{aligned}$$

Substituting $C_1 = -C_2$ in this last equation,

$$\begin{aligned} -1 &= e^{-6}(-6(-C_2) - 5C_2) \\ C_2 &= -e^6. \end{aligned}$$

Then, $C_1 = -C_2 = e^6$. Thus, the particular solution is

$$\begin{aligned} y(t) &= (e^6 - e^6 t)e^{-6t} \\ &= (1 - t)e^{-6(t-1)}. \end{aligned}$$

28.

29. If the equation $y'' + py' + qy = 0$ has characteristic equation $\lambda^2 + p\lambda + q = 0$, and the characteristic equation has a double root, $\lambda = \lambda_1$, then we know that $\lambda_1^2 + p\lambda_1 + q = 0$. Secondly, because

$$\begin{aligned} 0 &= \lambda^2 + p\lambda + q \\ &= (\lambda - \lambda_1)(\lambda - \lambda_1) \\ &= \lambda^2 - 2\lambda_1\lambda + \lambda_1^2, \end{aligned}$$

we can compare coefficients and note that $-2\lambda_1 = p$, or $2\lambda_1 + p = 0$. If $y(t) = te^{\lambda_1 t}$, then

$$\begin{aligned} y'(t) &= (\lambda_1 t + 1)e^{\lambda_1 t} \\ y''(t) &= (2\lambda_1 + \lambda_1^2 t)e^{\lambda_1 t}. \end{aligned}$$

Now,

$$\begin{aligned} y'' + py' + qy &= (2\lambda_1 + \lambda_1^2 t)e^{\lambda_1 t} + p(\lambda_1 t + 1)e^{\lambda_1 t} + qte^{\lambda_1 t} \\ &= e^{\lambda_1 t}(2\lambda_1 + p) + t(\lambda_1^2 + p\lambda_1 + q)e^{\lambda_1 t} \\ &= 0, \end{aligned}$$

where we've used the facts that $2\lambda_1 + p = 0$ and $\lambda_1^2 + p\lambda_1 + q = 0$. Thus, $y(t) = te^{\lambda_1 t}$ is a solution of $y'' + py' + qy = 0$.

30. (a)

(b)

(c)

(d)

31. By Euler's identity,

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{A}$$

Similarly,

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta. \tag{B}$$

Adding (A) and (B) produces

$$\begin{aligned} e^{i\theta} + e^{-i\theta} &= 2\cos\theta \\ \cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2}. \end{aligned}$$

Subtracting (B) from (A) gives

$$\begin{aligned} e^{i\theta} - e^{-i\theta} &= 2i\sin\theta \\ \sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i}. \end{aligned}$$

32.

33. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then,

$$\begin{aligned} e^{z_1+z_2} &= e^{(x_1+iy_1)+(x_2+iy_2)} \\ &= e^{(x_1+x_2)+i(y_1+y_2)} \\ &= e^{x_1+x_2}e^{i(y_1+y_2)} \\ &= e^{x_1}e^{x_2}(\cos(y_1+y_2) + i\sin(y_1+y_2)) \\ &= e^{x_1}e^{x_2}(\cos y_1\cos y_2 - \sin y_1\sin y_2 + i\sin y_1\cos y_2 + i\sin y_2\cos y_1) \\ &= e^{x_1}e^{x_2}(\cos y_1\cos y_2 + i\cos y_1\sin y_2 + i\sin y_1\cos y_2 + i^2\sin y_1\sin y_2) \\ &= e^{x_1}e^{x_2}[\cos y_1(\cos y_2 + i\sin y_2) + i\sin y_1(\cos y_2 + i\sin y_2)] \\ &= e^{x_1}e^{x_2}(\cos y_1 + i\sin y_1)(\cos y_2 + i\sin y_2) \\ &= e^{x_1}(\cos y_1 + i\sin y_1)e^{x_2}(\cos y_2 + i\sin y_2) \\ &= e^{x_1}e^{iy_1}e^{x_2}e^{iy_2} \\ &= e^{x_1+iy_1}e^{x_2+iy_2} \\ &= e^{z_1}e^{z_2}. \end{aligned}$$

34.

35. If $z = x + iy$, then

$$\begin{aligned}\overline{e^z} &= \overline{e^{x+iy}}\\ &= \overline{e^x e^{iy}}\\ &= \overline{e^x(\cos y + i\sin y)}\\ &= e^x(\cos y - i\sin y)\\ &= e^x(\cos(-y) + i\sin(-y))\\ &= e^x e^{-iy}\\ &= e^{x-iy}\\ &= e^{\bar{z}}\end{aligned}$$

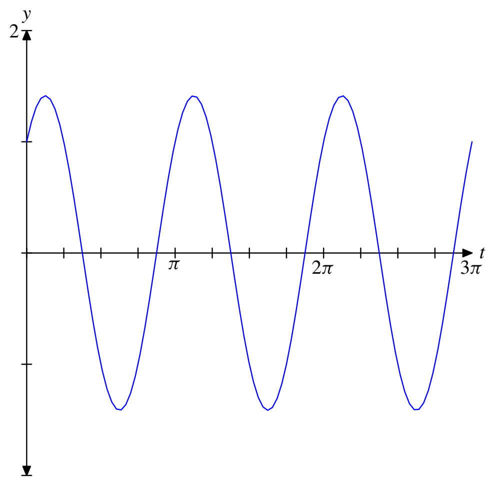
36.

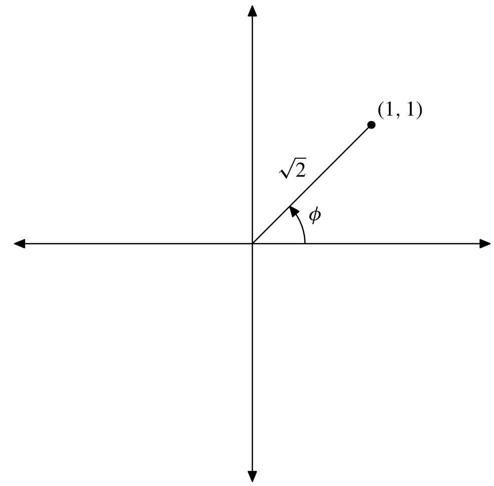
37. If $\lambda = a + bi$, then

$$\begin{aligned}\frac{d}{dt}\left\{e^{\lambda t}\right\} &= \frac{d}{dt}\left\{e^{(a+bi)t}\right\}\\ &= \frac{d}{dt}\left\{e^{at}e^{ibt}\right\}\\ &= \frac{d}{dt}\left\{e^{at}(\cos bt + i\sin bt)\right\}\\ &= \frac{d}{dt}\left\{e^{at}\cos bt + ie^{at}\sin bt\right\}\\ &= \frac{d}{dt}e^{at}\cos bt + i\frac{d}{dt}e^{at}\sin bt\\ &= e^{at}(-b\sin bt) + ae^{at}\cos bt + ie^{at}b\cos bt + iae^{at}\sin bt\\ &= e^{at}(a\cos bt + ia\sin bt + ib\cos bt + i^2 b\sin bt)\\ &= e^{at}(a(\cos bt + i\sin bt) + ib(\cos bt + i\sin bt))\\ &= e^{at}(a+ib)(\cos bt + i\sin bt)\\ &= (a+ib)e^{at}e^{ibt}\\ &= (a+ib)e^{(a+ib)t}\\ &= \lambda e^{\lambda t}.\end{aligned}$$

Section 4.4

1. The first of two images contains the plot of $y = \cos 2t + \sin 2t$ on the interval $[0, 3\pi]$.





Plot the coefficients of

$$y = 1 \cdot \cos 2t + 1 \cdot \sin 2t$$

in the first quadrant, calculate the magnitude of the vector, and mark the angle. The angle ϕ is easily calculated.

$$\begin{aligned} \tan\phi &= \frac{1}{1} = 1 \\ \phi &= \frac{\pi}{4}. \end{aligned}$$

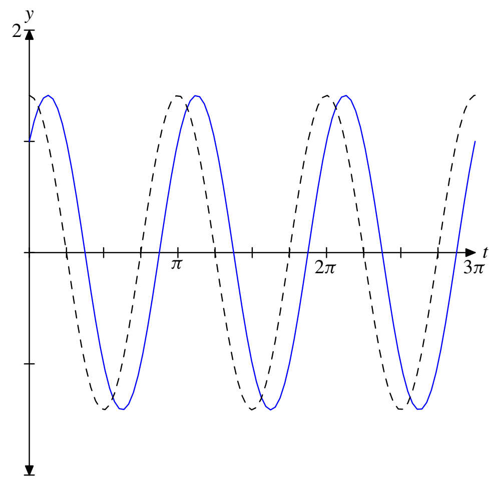
Factor out the magnitude $\sqrt{2}$ as follows.

$$y = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos 2t + \frac{1}{\sqrt{2}}\sin 2t\right)$$

But $\cos\phi = 1/\sqrt{2}$ and $\sin\phi = 1/\sqrt{2}$, so we can write

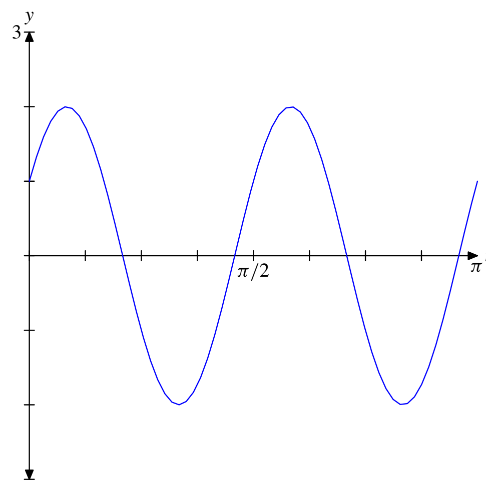
$$\begin{aligned} y &= \sqrt{2}\,(\cos\phi\cos 2t + \sin\phi\sin 2t) \\ y &= \sqrt{2}\cos(2t - \phi) \\ y &= \sqrt{2}\cos\left(2t - \frac{\pi}{4}\right) \\ y &= \sqrt{2}\cos 2\left(t - \frac{\pi}{8}\right). \end{aligned}$$

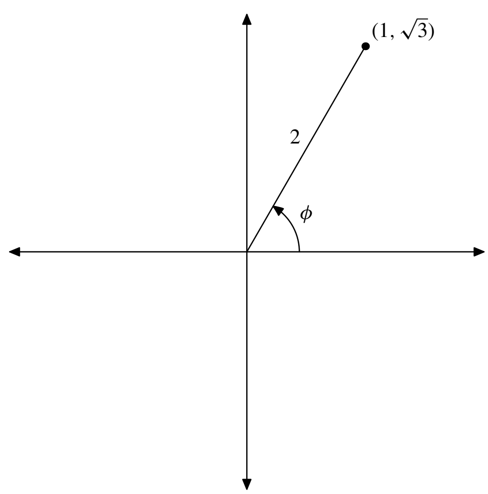
Hence, the curve has amplitude $\sqrt{2}$, period $T = \pi$, and is shifted to the right $\pi/8$ units. This is clearly shown in the following image where the dashed curve is the unshifted $y = \sqrt{2}\cos 2t$.



2.

3. The first of two images contains the plot of $y = \cos 4t + \sqrt{3} \sin 4t$ on the interval $[0, \pi]$.





Plot the coefficients of

$$y = 1 \cdot \cos 4t + \sqrt{3} \cdot \sin 4t$$

in the first quadrant, calculate the magnitude of the vector, and mark the angle. The angle ϕ is easily calculated.

$$\tan \phi = \frac{\sqrt{3}}{1} = \sqrt{3}$$
$$\phi = \frac{\pi}{3}.$$

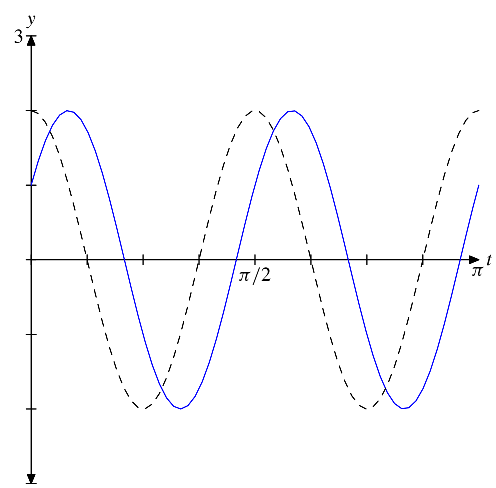
Factor out the magnitude 2 as follows.

$$y = 2\left(\frac{1}{2}\cos 4t + \frac{\sqrt{3}}{2}\sin 4t\right)$$

But $\cos\phi = 1/2$ and $\sin\phi = \sqrt{3}/2$, so we can write

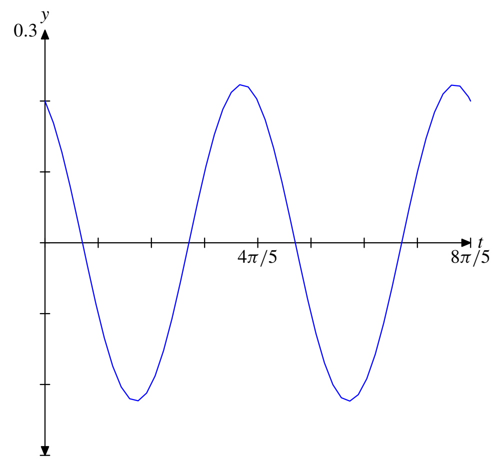
$$\begin{aligned}
y &= 2\left(\cos\phi\cos 4t + \sin\phi\sin 4t\right)\\
y &= 2\cos(4t - \phi)\\
y &= 2\cos\left(4t - \frac{\pi}{3}\right)\\
y &= 2\cos 4\left(t - \frac{\pi}{12}\right).
\end{aligned}$$

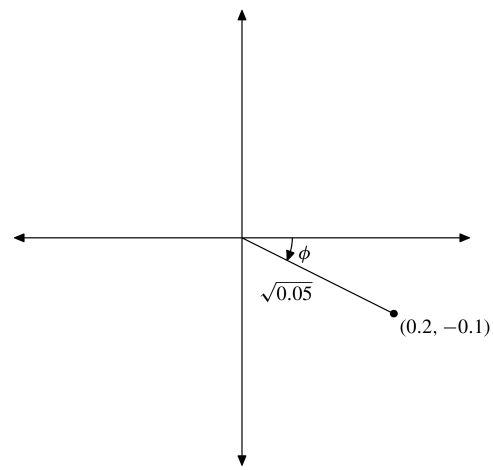
Hence, the curve has amplitude 2, period $T = \pi/2$, and is shifted to the right $\pi/12$ units. This is clearly shown in the following image where the dashed curve is the unshifted $y = 2\cos 4t$.



4.

5. The first of the two images imges contains the plot of $y = 0.2\cos 2.5t - 0.1\sin 2.5t$ on the interval $[0, 8\pi/5]$.





Plot the coefficients of

$$y = 0.2\cos 2.5t - 0.1\sin 2.5t$$

in the fourth quadrant, calculate the magnitude of the vector, and mark the angle. The angle ϕ is easily calculated.

$$\tan\phi = \frac{-1}{2} = -\frac{1}{2}$$
$$\phi \approx -0.4636.$$

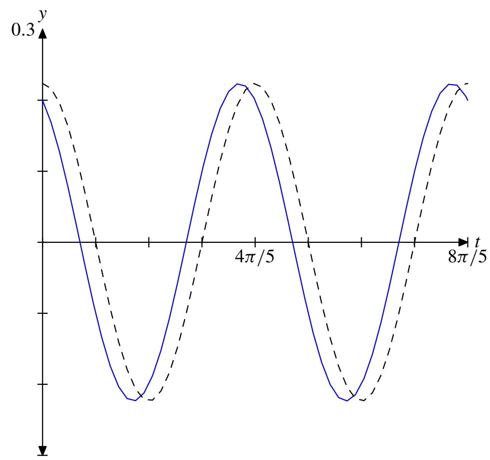
Factor out the magnitude 2 as follows.

$$y = \sqrt{0.05}\left(\frac{0.2}{\sqrt{0.05}}\cos 2.5t - \frac{0.1}{\sqrt{0.05}}\sin 2.5t\right)$$

But $\cos\phi = 0.2/\sqrt{0.05}$ and $\sin\phi = -0.1/\sqrt{0.05}$, so we can write

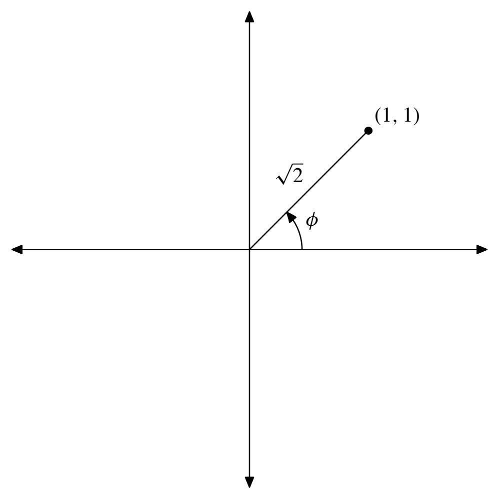
$$y = \sqrt{0.05}\,(\cos\phi\cos 2.5t + \sin\phi\sin 2.5t)$$
$$y = 0.2236\cos(2.5t - \phi)$$
$$y = 0.2236\cos(2.5t + 0.4636)$$
$$y = .2236\cos 2.5\,(t + 0.1855)\,.$$

Hence, the curve has amplitude 0.2236, period $T = 2\pi/2.5 = 4\pi/5$, and is shifted to the left 0.1855 units. This is clearly shown in the second image above, where the dashed curve is the unshifted $y = 0.2236\cos 2.5t$.



6.

7. Plot the coefficients of $1 \cdot \cos 5t + 1 \cdot \sin 5t$ in the first quadrant, calculate the magnitude of the vector, and mark the angle.



The angle ϕ is easily calculated.

$$\tan\phi = 1$$
$$\phi = \frac{\pi}{4}$$

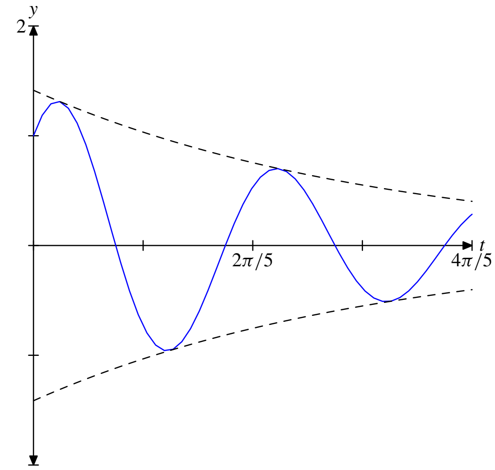
Factor out the magnitude as follows:

$$y = \sqrt{2}e^{-t/2}\left(\frac{1}{\sqrt{2}}\cos 5t + \frac{1}{\sqrt{2}}\sin 5t\right)$$

But $\cos\phi = 1/\sqrt{2}$ and $\sin\phi = 1/\sqrt{2}$, so we can write

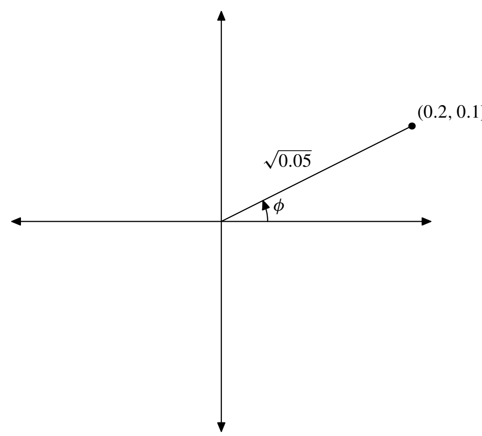
$$y = \sqrt{2}e^{-t/2}(\cos\phi\cos 5t + \sin\phi\sin 5t)$$
$$y = \sqrt{2}e^{-t/2}\cos(5t - \phi)$$
$$y = \sqrt{2}e^{-t/2}\cos(5t - \pi/4).$$

Thus, the amplitude is $\sqrt{2}e^{-t/2}$ which, along with $-\sqrt{2}e^{-t/2}$, clearly bound the graph of $y = \sqrt{2}e^{-t/2}\cos(5t - \pi/4)$.



8.

9. Plot the coefficients of $0.2 \cdot \cos 2t + 0.1 \cdot \sin 2t$ in the first quadrant, calculate the magnitude of the vector, and mark the angle.



The angle ϕ is easily calculated.

$$\tan\phi = \frac{0.1}{0.2} = \frac{1}{2}$$
$$\phi = 0.4636$$

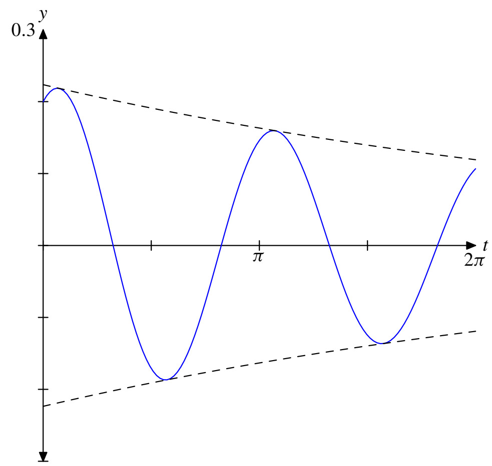
Factor out the magnitude as follows:

$$y = \sqrt{0.05}e^{-0.1t}\left(\frac{0.2}{\sqrt{0.05}}\cos 2t + \frac{0.1}{\sqrt{0.05}}\sin 2t\right)$$

But $\cos\phi = 0.2/\sqrt{0.05}$ and $\sin\phi = 0.1/\sqrt{0.05}$, so we can write

$$y = \sqrt{0.05}e^{-0.1t}\left(\cos\phi\cos 2t + \sin\phi\sin 2t\right)$$
$$y = 0.2236e^{-0.1t}\cos(2t - \phi)$$
$$y = 0.2236e^{-0.1}\cos(2t - 0.4636).$$

Thus, the amplitude is $0.2236e^{-0.1t}$ which, along with $-0.2236e^{-0.1t}$, clearly bound the graph of $y = 0.2236e^{-0.1t}\cos(2t - 0.4636)$.



10.

11. Substitute $m = 0.2$ kg and $k = 5\text{ kg/s}^2$ in $my'' + ky = 0$ to obtain $0.2y'' + 5y = 0$ or

$$y'' + 25y = 0.$$

The characteristic equation is $\lambda^2 + 25 = 0$, with zeros $\lambda = \pm 5i$, so

$$z(t) = e^{5it} = \cos 5t + i\sin 5t$$

is a complex solution. The real and imaginary parts of this solution form a fundamental set of real solutions, giving the general solution

$$y(t) = C_1\cos 5t + C_2\sin 5t.$$

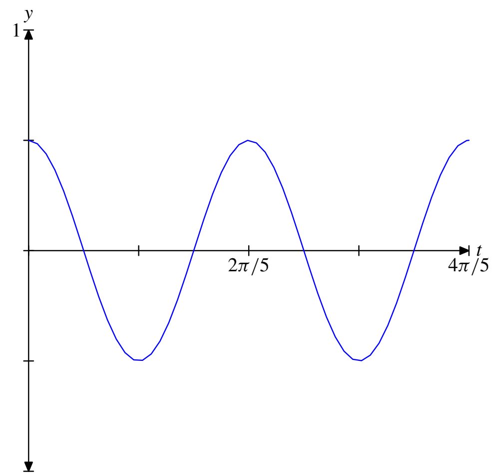
The initial displacement is 0.5 m, so $y(0) = 0.5$ and $C_1 = 0.5$. Differentiating,

$$y'(t) = -5C_1\sin 5t + 5C_2\cos 5t.$$

The system is released from rest, so $y'(0) = 0$ and $C_2 = 0$. Thus, the solution is

$$y(t) = 0.5\cos 5t,$$

which has amplitude 0.5, frequency 5 rad/s, and zero phase.



12.

13. The system $(2/5)x'' + kx = 0$, or $x'' + (5k/2)x = 0$, is equivalent to

$$x'' + \omega_0^2 x = 0,$$

with $\omega_0^2 = 5k/2$. The characteristic equation is $\lambda^2 + \omega_0^2 = 0$ with roots $\lambda = \pm\omega_0 i$. Thus, we have complex solution

$$z(t) = e^{i\omega_0 t} = \cos\omega_0 t + i\sin\omega_0 t.$$

The real and imaginary parts of this solution form a fundamental set of solutions and provide the general solution

$$x(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t.$$

This solution is periodic with period $T = 2\pi/\omega_0$. Thus, if it is know that the period is $\pi/2$, then

$$\begin{aligned} T &= \frac{2\pi}{\omega_0} \\ \frac{\pi}{2} &= \frac{2\pi}{\omega_0} \\ \omega_0 &= 4. \end{aligned}$$

But $\omega_0^2 = 5k/2$,

$$\begin{aligned} 4^2 &= \frac{5k}{2} \\ k &= \frac{32}{5}. \end{aligned}$$

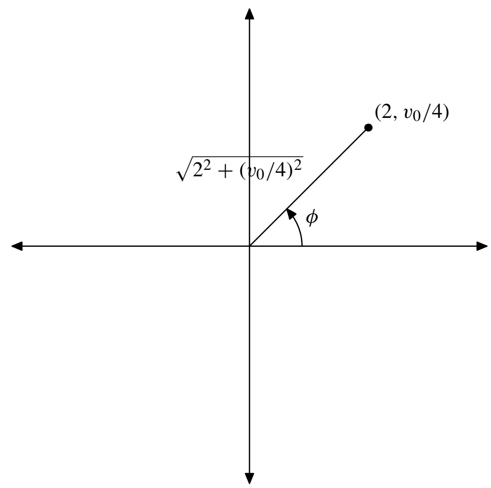
The initial condition $x(0) = 2$ gives $C_1 = 2$. Differentiating $x(t)$,

$$x'(t) = -C_1\omega_0\sin\omega_0 t + C_2\omega_0\cos\omega_0 t.$$

The initial condition $x'(0) = v_0$ leads to $C_2 = v_0/\omega_0$ and the solution

$$x(t) = 2\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t = 2\cos 4t + \frac{v_0}{4}\sin 4t.$$

Plot the coefficients, calculate the magnitude of the vector, and mark the angle.



As in Exercises 1–6, factoring out the magnitude will place the solution in the form

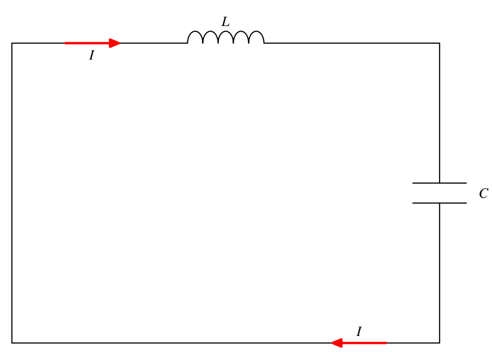
$$x(t) = \sqrt{4 + v_0^2/16}\cos(4t - \phi),$$

where $\tan\phi = v_0/8$. If it is known that the amplitude is 2, then

$$\sqrt{4 + v_0^2/16} = 2$$
$$4 + v_0^2/16 = 4$$
$$v_0 = 0.$$

14.

15. (a) In an LC circuit,



the sum of the voltage drops around the closed loop is zero. The voltage drop across the inductor is LI', where $L = 6\,\mu$H and $I(t)$ is the current, and the voltage drop across the capacitor is $(1/C)Q$, where $C = 2\,\mu$F and $Q(t)$ is the charge on the capacitor. Thus, we can write

$$LI' + \frac{1}{C}Q = 0,$$

or, since $I = dQ/dt$,

$$LQ'' + \frac{1}{C}Q = 0.$$

To find the initial charge on the capacitor, use

$$\begin{aligned} V_C &= \frac{1}{C}Q \\ Q &= CV_C \\ Q &= (2 \times 10^{-6}\,\text{F})(20\,V) \\ Q &= 4.0 \times 10^{-5}\text{coulombs}. \end{aligned}$$

(b) If the initial current is zero, then $Q'(0) = I(0) = 0$. Thus, $LQ'' + (1/C)Q = 0$, or

$$Q'' + \frac{1}{LC}Q = 0, \qquad Q(0) = 4.0 \times 10^{-5},\ Q'(0) = 0.$$

This last equation is equivalent to

$$Q'' + \omega_0^2 Q = 0,$$

where $\omega_0^2 = 1/(LC)$. This has characteristic equation $\lambda^2 + \omega_0^2 = 0$ and zeros $\lambda = \pm i\omega_0$, leading to the complex solution

$$z(t) = e^{i\omega_0 t} = \cos\omega_0 t + i\sin\omega_0 t.$$

The real and imaginary parts of this solution form a fundamental set of solutions, leading to the general solution

$$Q(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t.$$

The initial condition $Q(0) = 4.0 \times 10^{-5}$ gives $C_1 = 4.0 \times 10^{-5}$. Differentiating $Q(t)$,

$$Q'(t) = -C_1\omega_0\sin\omega_0 t + C_2\omega_0\cos\omega_0 t.$$

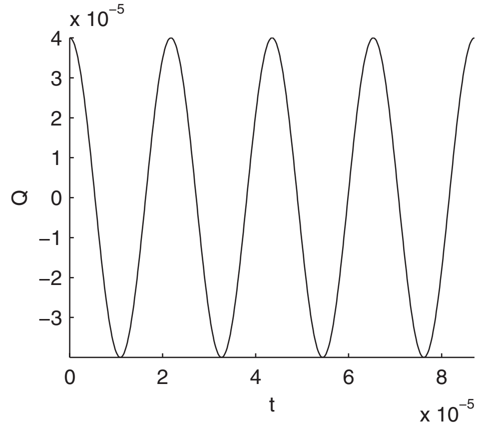
The initial condition $Q'(0) = 0$ gives $C_2 = 0$ and the solution

$$Q(t) = (4.0 \times 10^{-5})\cos\omega_0 t.$$

Thus, the amplitude is 4.0×10^{-5}. Because $\omega_0^2 = 1/(LC)$,

$$\begin{aligned} \omega_0 &= \sqrt{\frac{1}{LC}} \\ &= \sqrt{\frac{1}{(6 \times 10^{-6})(2 \times 10^{-6})}} \\ &\approx 2.887 \times 10^5\ \text{rad/s}. \end{aligned}$$

The phase is zero.



16. (a)
(b)

17. Suppose that the spring-mass system $my'' + \mu y' + ky = 0$ is overdamped. Then,

$$y'' + \frac{\mu}{m}y' + \frac{k}{m}y = 0$$
$$y'' + 2cy' + \omega_0^2 y = 0,$$

where $2c = \mu/m$ and $\omega_0^2 = k/m$. The characteristic equation is $\lambda^2 + 2c\lambda + \omega_0^2 = 0$ and the zeros are

$$\lambda_1 = -c - \sqrt{c^2 - \omega_0^2} \qquad \text{and} \qquad \lambda_2 = -c + \sqrt{c^2 - \omega_0^2}.$$

In order that the system be overdamped, we need $c^2 - \omega_0^2 > 0$. Of course, this condition results in $\lambda_1 < \lambda_2 < 0$ and the general solution is

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

To determine how many times this solution can cross the t-axis, set $y(t) = 0$ and solve for t.

$$0 = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$
$$0 = e^{\lambda_1 t}(C_1 + C_2 e^{(\lambda_2 - \lambda_1)t})$$

Of course, $e^{\lambda_1 t}$ is never zero, so this leads to

$$C_1 + C_2 e^{(\lambda_2 - \lambda_1)t} = 0$$
$$e^{(\lambda_2 - \lambda_1)t} = -\frac{C_1}{C_2}.$$

If $C_1/C_2 > 0$, then there are no crossings, but if $C_1/C_2 < 0$, there will be exactly one crossing of the t-axis. The challenge is to find initial conditions y_0 and v_0 that lead to zero and one crossings. But $y(0) = y_0$ gives

$$y_0 = C_1 + C_2.$$

Differentiating $y(t)$,

$$y'(t) = C_1 \lambda_1 e^{\lambda_1 t} + C_2 \lambda_2 e^{\lambda_2 t},$$

the initial condition $y'(0) = v_0$ provides

$$v_0 = C_1\lambda_1 + C_2\lambda_2,$$

and this system is easily solved for C_1 and C_2,

$$C_1 = \frac{\lambda_2 y_0 - v_0}{\lambda_2 - \lambda_1} \qquad \text{and} \qquad C_2 = \frac{v_0 - \lambda_1 y_0}{\lambda_2 - \lambda_1},$$

which gives us

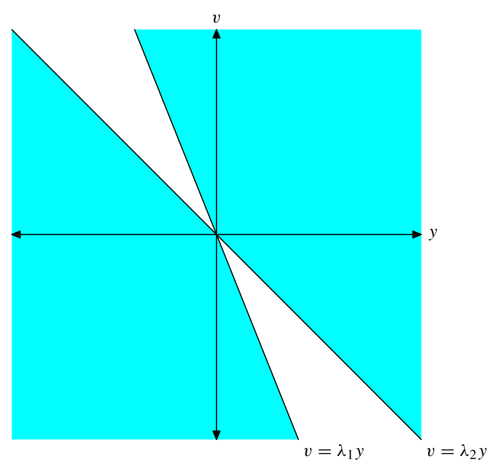
$$\frac{C_1}{C_2} = \frac{\lambda_2 y_0 - v_0}{v_0 - \lambda_1 y_0}.$$

Because there is exactly one zero crossing if $C_1/C_2 < 0$, we see two possible ways that this can happen:

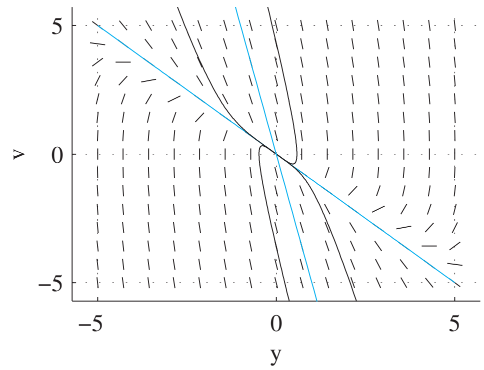
Case 1: We must have $\lambda_2 y_0 - v_0 < 0$ and $v_0 - \lambda_1 y_0 > 0$, which leads to the condition that $v_0 > \lambda_2 y_0$ and $v_0 > \lambda_1 y_0$, or

Case 2: We must have $\lambda_2 y_0 - v_0 > 0$ and $v_0 - \lambda_1 y_0 < 0$, which leads to the condition that $v_0 < \lambda_2 y_0$ and $v_0 < \lambda_1 y_0$.

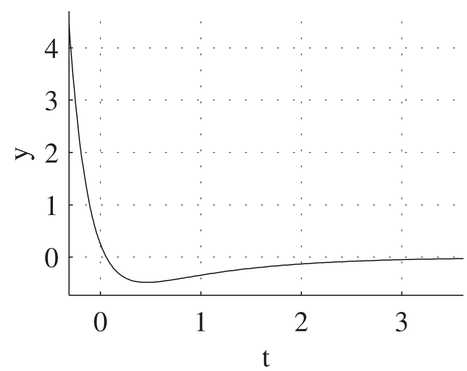
These conditions are easy to analyze if you think geometrically.



Thus, there will be one zero crossing if you select an initial condition from the shaded region. This is not especially intuitive until you give it a try in your solver. For example, the spring mass system $y'' + 6y'' + 5y = 0$, with characteristic equation $\lambda^2 + 6\lambda + 5 = 0$ and zeros $\lambda_1 = -5$ and $\lambda_2 = -1$ is overdamped. If we sketch the lines $v = -5y$ and $v = -y$ on our phase portrait, note, that as predicted, those initial conditions starting in the shaded region cross $y = 0$ exactly once.



Of course, what we really want is a plot of y versus t.



Note that the graph crosses the t-axis exactly once. Finally, by picking initial conditions from the unshaded region, you can find a solution that does not cross the time axis.

Further refinement of the above arguments is possible. For example, how would you limit the initial conditions so that the solution crosses at some time $t > 0$?

18.

19. Suppose that the spring-mass system $my'' + \mu y' + ky = 0$ is critically damped. Then,

$$y'' + \frac{\mu}{m}y' + \frac{k}{m}y = 0$$
$$y'' + 2cy' + \omega_0^2 y = 0,$$

where $2c = \mu/m$ and $\omega_0^2 = k/m$. The characteristic equation is $\lambda^2 + 2c\lambda + \omega_0^2 = 0$ and the zeros are

$$\lambda_1 = -c - \sqrt{c^2 - \omega_0^2} \qquad \text{and} \qquad \lambda_2 = -c + \sqrt{c^2 - \omega_0^2}.$$

In order that the system be critically damped, we need $c^2 - \omega_0^2 = 0$. Of course, this condition results in $\lambda_1 = \lambda_2 = -c$ and the general solution is

$$y(t) = (C_1 + C_2 t)e^{-ct}.$$

To determine how many times this solution can cross the t-axis, set $y(t) = 0$ and solve for t.

$$0 = (C_1 + C_2 t)e^{-ct}$$

Of course, e^{-ct} is never zero, so this leads to

$$\begin{aligned} C_1 + C_2 t &= 0 \\ t &= -\frac{C_1}{C_2}. \end{aligned}$$

Thus, the solution will always cross the t-axis, but let's find only those crossings when $t > 0$, which necessitates that $C_1/C_2 < 0$. The challenge is to find initial conditions y_0 and v_0 that lead to a crossing when $t > 0$. But $y(0) = y_0$ gives

$$y_0 = C_1.$$

Differentiating $y(t)$,

$$y'(t) = ((C_2 - cC_1) - cC_2 t)e^{-ct},$$

the initial condition $y'(0) = v_0$ provides

$$v_0 = C_2 - cC_1,$$

and this system is easily solved for C_1 and C_2,

$$C_1 = y_0 \qquad \text{and} \qquad C_2 = v_0 + cy_0,$$

which gives us

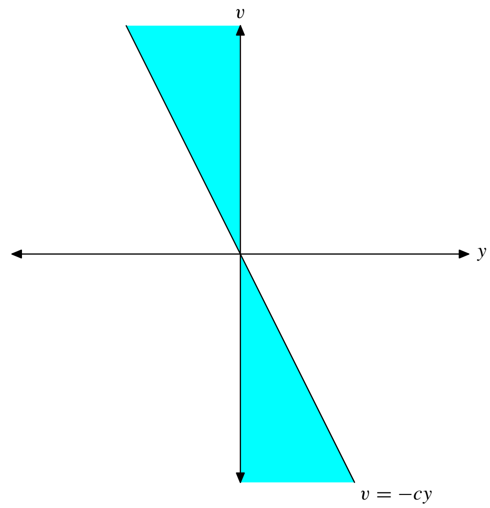
$$\frac{C_1}{C_2} = \frac{y_0}{v_0 + cy_0}.$$

Because there is a zero crossing at $t > 0$ if $C_1/C_2 < 0$, we see two possible ways that this can happen:

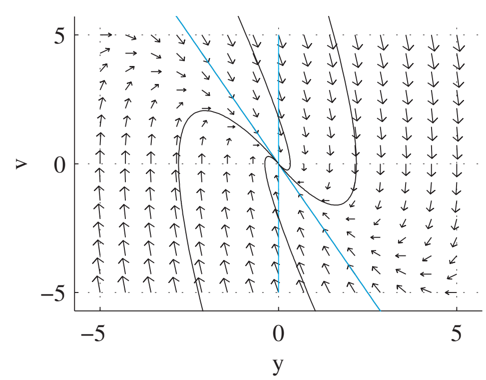
Case 1: We must have $y_0 > 0$ and $v_0 + cy_0 < 0$, which leads to the condition that $y_0 > 0$ and $v_0 < -cy_0$, or

Case 2: We must have $y_0 < 0$ and $v_0 + cy_0 > 0$, which leads to the condition that $y_0 < 0$ and $v_0 > -cy_0$.

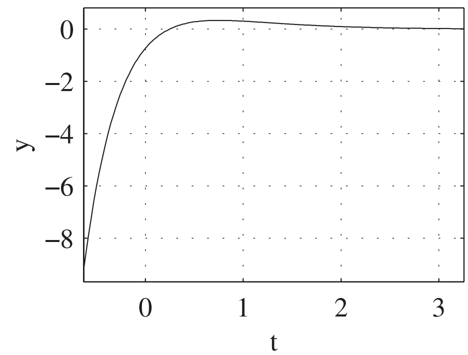
These conditions are easy to analyze if you think geometrically.



Thus, there will be one zero crossing if you select an initial condition from the shaded region. This is not especially intuitive until you give it a try in your solver. For example, the spring mass system $y'' + 4y'' + 4y = 0$, with characteristic equation $\lambda^2 + 4\lambda + 4 = 0$ and repeated zero $\lambda = -2$ is critically damped. If we sketch the lines $v = -2y$ on our phase portrait, note, that as predicted, those initial conditions starting in the shaded region cross $y = 0$ exactly once at $t > 0$.



Of course, what we really want is a plot of y versus t.



Note that the graph crosses the t-axis exactly once. Finally, by picking initial conditions from the unshaded region, you will note that this solution also crosses the y-axis exactly once, but at $t < 0$.

20. (a)
(b)

21. (a) Suppose that $mx'' + \mu x' + kx = 0$ is overdamped. We can write

$$\begin{aligned} x'' + \frac{\mu}{m}x' + \frac{k}{m}x &= 0 \\ x'' + 2cx' + \omega_0^2 &= 0, \end{aligned}$$

where $2c = \mu/m$ and $\omega_0^2 = k/m$. The system has characteristic equation $\lambda^2 + 2c\lambda + \omega_0^2 = 0$ and zeros

$$\lambda_1 = -c - \sqrt{c^2 - \omega_0^2} \qquad \text{and} \qquad \lambda_2 = -c + \sqrt{c^2 - \omega_0^2}.$$

If the system is overdamped, note that

$$\begin{aligned} c^2 - \omega_0^2 &> 0 \\ \left(\frac{\mu}{2m}\right)^2 &> \frac{k}{m} \\ \mu^2 &> 4mk \\ \mu &> 2\sqrt{mk}. \end{aligned}$$

and the general solution is

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

The initial condition $x(0) = 0$ gives $0 = C_1 + C_2$ and $C_1 = -C_2$. Differentiating $x(t)$,

$$x'(t) = C_1\lambda_1 e^{\lambda_1 t} + C_2\lambda_2 e^{\lambda_2 t},$$

and the initial condition $x'(0) = v_0$ provides $v_0 = C_1\lambda_1 + C_2\lambda_2$. This system is easily solved for

$$C_1 = \frac{v_0}{\lambda_1 - \lambda_2} \qquad \text{and} \qquad C_2 = \frac{-v_0}{\lambda_1 - \lambda_2},$$

so

$$x(t) = \frac{v_0}{\lambda_1 - \lambda_2}(e^{\lambda_1 t} - e^{\lambda_2 t}).$$

But,

$$\lambda_1 - \lambda_2 = -2\sqrt{c^2 - \omega_0^2},$$

and

$$\begin{aligned} x(t) &= -\frac{v_0}{2\sqrt{c^2-\omega_0^2}}\left(e^{\left(-c-\sqrt{c^2-\omega_0^2}\right)t} - e^{\left(-c+\sqrt{c^2-\omega_0^2}\right),t}\right) \\ &= \frac{v_0}{\sqrt{c^2-\omega_0^2}}e^{-ct}\left(\frac{e^{\sqrt{c^2-\omega_0^2}\,t} - e^{-\sqrt{c^2-\omega_0^2}\,t}}{2}\right) \\ &= \frac{v_0}{\sqrt{c^2-\omega_0^2}}e^{-ct}\sinh\sqrt{c^2-\omega_0^2}\,t. \end{aligned}$$

But, $c = \mu/(2m)$ and

$$\begin{aligned} \sqrt{c^2-\omega_0^2} &= \sqrt{\frac{\mu^2}{4m^2} - \frac{k}{m}} \\ &= \sqrt{\frac{\mu^2 - 4km}{4m^2}} \\ &= \frac{\sqrt{\mu^2-4km}}{2m}. \end{aligned}$$

Hence,

$$x(t) = \frac{v_0}{\gamma}e^{-\mu t/(2m)}\sinh\gamma t,$$

where $\gamma = \sqrt{c^2-\omega_0^2} = \sqrt{\mu^2-4km}/(2m)$.

(b) The initial position is zero. The initial velocity is in the downward direction. Hence, due to the fact that the system is overdamped, we know that the solution has already crossed the t-axis once, and the motion is downward, after which the solution will approach the t-axis asymptotically. Therefore, the mass will reach its lowest point the *first* time the velocity is zero. Differentiating $x(t)$,

$$\begin{aligned} v(t) &= x'(t) \\ &= \frac{v_0}{\gamma}e^{-\mu t/(2m)}(\cosh\gamma t)(\gamma) + \frac{v_0}{\gamma}\left(-\frac{\mu}{2m}\right)e^{-\mu t/(2m)}\sinh\gamma t \\ &= v_0 e^{\mu t/(2m)}\left(\cosh\gamma t - \frac{\mu}{2m\gamma}\sinh\gamma t\right). \end{aligned}$$

Setting the velocity equal to zero, it must be the case that

$$\begin{aligned}\cosh\gamma t - \frac{\mu}{2m\gamma}\sinh\gamma t &= 0\\ \frac{\mu}{2m\gamma}\sinh\gamma t &= \cosh\gamma t\\ \tanh\gamma t &= \frac{2m\gamma}{\mu}.\end{aligned}$$

Taking the inverse hyperbolic tangent,

$$\begin{aligned}\gamma t &= \tanh^{-1}\left(\frac{2m\gamma}{\mu}\right)\\ t &= \frac{1}{\gamma}\tanh^{-1}\left(\frac{2m\gamma}{\mu}\right).\end{aligned}$$

(c) If the system is critically damped, then $c^2 - \omega_0^2 = 0$ and the characteristic equation produces a repeated zero, $\lambda = -c$. Thus, the general solution is

$$x(t) = (C_1 + C_2 t)e^{-ct}.$$

The initial condition $x(0) = 0$ produces $C_1 = 0$. The derivative of $x(t)$ is

$$\begin{aligned}v(t) &= x'(t)\\ &= (C_1 + C_2 t)(-ce^{-ct}) + C_2 e^{-ct}\\ &= ((C_2 - cC_1) - cC_2 t)e^{-ct}.\end{aligned}$$

The initial condition $x'(0) = v_0$ gives $v_0 = C_2 - cC_1$, so the fact that $C_1 = 0$ produces $C_2 = v_0$ and the solution

$$x(t) = v_0 t e^{-ct}$$

But $c = \mu/(2m)$, so

$$x(t) = v_0 t e^{-\mu t/(2m)}$$

and

$$\begin{aligned}v(t) &= x'(t)\\ &= v_0 t\left(-\frac{\mu}{2m}\right)e^{-\mu t/(2m)} + v_0 e^{-\mu t/(2m)}\\ &= v_0 e^{-\mu t/(2m)}\left(-\frac{\mu t}{2m} + 1\right).\end{aligned}$$

Because the system is critically damped, and the solution has already crossed the t-axis, the solution will travel downward, then approach the t-axis asymptotically. Hence, the solution reaches its lowest point when the velocity *first* equals zero. This occurs when

$$\begin{aligned}\frac{\mu t}{2m} &= 1\\ t &= \frac{2m}{\mu}.\end{aligned}$$

22.

23. The voltage across the capacitor is given by $V_C = (1/C)Q$. Because the initial voltage across the capacitor is $V_c(0) = 50\,\text{V}$,

$$Q(0) = CV_C(0) = (0.008\,\text{F})(50\,\text{V}) = 0.4\,\text{coulombs}.$$

The sum of the voltage drops around the LRC circuit equals zero, so $LI' + RI + (1/C)Q = 0$. Because $I = Q'$, this equation becomes $LQ'' + RQ' + (1/C)Q = 0$. Substituting $L = 4\,\text{H}$, $R = 20\,\Omega$, and $C = 0.008\,\text{F}$, we get $4Q'' + 20Q' + 125Q = 0$. With no initial current this equation becomes

$$Q'' + 5Q' + \frac{125}{4}Q = 0, \qquad Q(0) = 0.4,\ Q'(0) = I(0) = 0.$$

The characteristic equation is $\lambda^2 + 5\lambda + (125/4) = 0$ with zeros $\lambda = -5/2 \pm 5i$. This leads to the complex solution

$$z(t) = e^{(-5/2+5i)t} = e^{-5t/2}(\cos 5t + i \sin 5t).$$

The real and imaginary parts of this solution provide a fundamental set of solutions and the general solution

$$Q(t) = e^{-5t/2}(C_1 \cos 5t + C_2 \sin 5t).$$

The initial condition $Q(0) = 0.4$ gives $C_1 = 2/5$. Differentiating $Q(t)$,

$$Q'(t) = e^{-5t/2}(-5C_1 \sin 5t + 5C_2 \cos 5t) - \frac{5}{2}e^{-5t/2}(C_1 \cos 5t + C_2 \sin 5t).$$

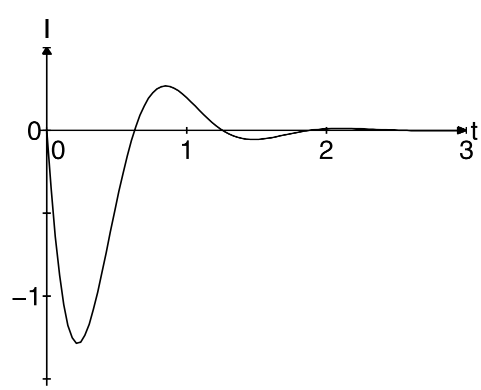
The initial condition $Q'(0) = 0$ leads to $0 = 5C_2 - (5/2)C_1$, and because $C_1 = 2/5$, we have $C_2 = 1/5$ and the solution

$$Q(t) = \frac{1}{5}e^{-5t/2}(2\cos 5t + \sin 5t).$$

We find the current by differentiating Q to get

$$I(t) = -\frac{5}{2}e^{-5t/2}\sin 5t.$$

Since $-\sin 5t = \cos(5t + \pi/2)$, we see that the phase is $\phi = -\pi/2$. Clearly the amplitude is $5/2$, and the frequency is 5 rad/s.



24.

25. The voltage across the capacitor is given by $V_C = (1/C)Q$. Because the initial voltage across the capacitor is $V_c(0) = 1\,\text{V}$,

$$Q(0) = CV_C(0) = (0.02\,\text{F})(1\,\text{V}) = 0.02\,\text{coulombs}.$$

The sum of the voltage drops around the LRC circuit equals zero, so $LI' + RI + (1/C)Q = 0$. Because $I = Q'$, this equation becomes $LQ'' + RQ' + (1/C)Q = 0$. Substituting $L = 10\,\text{H}$, $R = 40\,\Omega$, and $C = 0.02\,\text{F}$, we get $10Q'' + 40Q' + 50Q = 0$. With no initial current this equation becomes

$$Q'' + 4Q' + 5Q = 0, \qquad Q(0) = 0.02,\;\; Q'(0) = I(0) = 0.$$

The characteristic equation is $\lambda^2 + 4\lambda + 5 = 0$ with zeros $\lambda = -2 \pm i$. This leads to the complex solution

$$z(t) = e^{(-2+i)t} = e^{-2t}(\cos t + i \sin t).$$

The real and imaginary parts of this solution provide a fundamental set of solutions and the general solution

$$Q(t) = e^{-2t}(C_1 \cos t + C_2 \sin t).$$

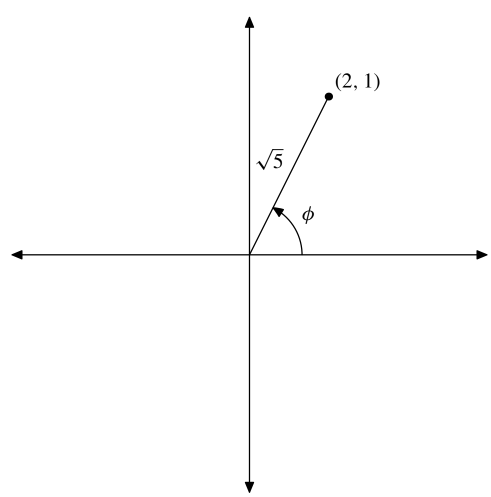
The initial condition $Q(0) = 0.02$ gives $C_1 = 1/50$. Differentiating $Q(t)$,

$$Q'(t) = e^{-2t}(-C_1 \sin t + C_2 \cos t) - 2e^{-2t}(C_1 \cos t + C_2 \sin t).$$

The initial condition $Q'(0) = 0$ leads to $0 = C_2 - 2C_1$, and because $C_1 = 1/50$, we have $C_2 = 1/25$ and the solution

$$Q(t) = \frac{1}{50}e^{-2t}(\cos t + 2 \sin t).$$

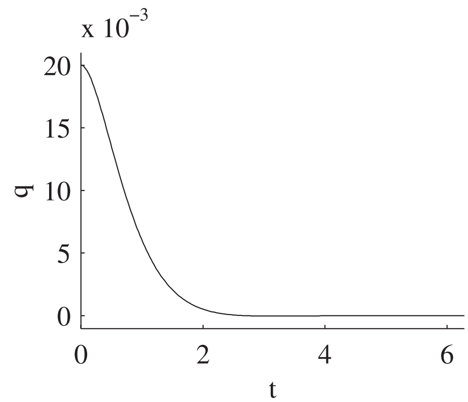
Plotting the coefficients of $\cos t$ and $\sin t$ and marking the angle,



we see that $\tan\phi = 2$, so $\phi = \arctan 2 \approx 1.1071$. Factoring out the magnitude of the radial vector,

$$\begin{aligned} Q(t) &= \frac{\sqrt{5}}{50}e^{-2t}\left(\frac{1}{\sqrt{5}}\cos t + \frac{2}{\sqrt{5}}\sin t\right) \\ &= \frac{\sqrt{5}}{50}e^{-2t}(\cos\phi\cos t + \sin\phi\sin t) \\ &= \frac{\sqrt{5}}{50}e^{-2t}\cos(t - 1.1071). \end{aligned}$$

Thus, the amplitude is $\sqrt{5}/50$, the frequency is 1 rad/s, and the phase is 1.1071.



Section 4.5

1. Let $y(t) = Ae^{-3t}$. Then

$$\begin{aligned} y'(t) &= -3Ae^{-3t} \\ y''(t) &= 9Ae^{-3t}, \end{aligned}$$

and $y'' + 3y' + 2y = 4e^{-3t}$ becomes

$$\begin{aligned} 9Ae^{-3t} + 3(-3Ae^{-3t}) + 2(Ae^{-3t}) &= 4e^{-3t} \\ 2A &= 4 \\ A &= 2. \end{aligned}$$

Thus, $y = 2e^{-3t}$ is a particular solution.

2.

3. Let $y(t) = Ae^{-t}$. Then

$$\begin{aligned} y'(t) &= -Ae^{-t} \\ y''(t) &= Ae^{-t}, \end{aligned}$$

and $y'' + 2y' + 5y = 12e^{-t}$ becomes

$$\begin{aligned} Ae^{-t} + 2(-Ae^{-t}) + 5(Ae^{-t}) &= 12e^{-t} \\ 4A &= 12 \\ A &= 3. \end{aligned}$$

Thus, $y = 3e^{-t}$ is a particular solution.

4.

5. Let $y_p = a\cos 3t + b\sin 3t$. Then

$$\begin{aligned} y_p' &= -3a\sin 3t + 3b\cos 3t \\ y_p'' &= -9a\cos 3t - 9b\sin 3t, \end{aligned}$$

and the equation $y'' + 4y = \cos 3t$ becomes

$$\begin{aligned} -9a\cos 3t - 9b\sin 3t + 4(a\cos 3t + b\sin 3t) &= \cos 3t \\ -5a\cos 3t - 5b\sin 3t &= \cos 3t. \end{aligned}$$

Thus, $-5a = 1$ and $-5b = 0$ lead to $a = -1/5$ and $b = 0$ and the particular solution $y_p = -1/5\cos 3t$.

6.

7. Let $y_p = a\cos 2t + b\sin 2t$. Then

$$\begin{aligned} y_p' &= -2a\sin 2t + 2b\cos 2t \\ y_p'' &= -4a\cos 2t - 4b\sin 2t, \end{aligned}$$

and substituting these results into $y'' + 7y' + 6y = 3\sin 2t$ leads to

$$(2a + 14b)\cos 2t + (-14a + 2b)\sin 2t = 3\sin 2t.$$

Thus,

$$\begin{aligned} 2a + 14b &= 0 \\ -14a + 2b &= 3, \end{aligned}$$

leading to $a = -21/100$ and $b = 3/100$. Thus, the particular solution is $y_p = -(21/100)\cos 2t + (3/100)\sin 2t$.

8.

9. If $z(t) = x(t) + iy(t)$, then

$$\begin{aligned} z'' + pz' + qz &= Ae^{i\omega t} \\ (x'' + iy'') + p(x' + iy') + q(x + iy) &= A(\cos\omega t + i\sin\omega t) \\ (x'' + px' + qx) + i(y'' + py' + qy) &= A\cos\omega t + iA\sin\omega t. \end{aligned}$$

Comparing real and imaginary parts,

$$\begin{aligned} x'' + px' + qx &= A\cos\omega t \\ y'' + py' + qy &= A\sin\omega t. \end{aligned}$$

Therefore, if $z(t) = x(t) + iy(t)$ is a solution of $z'' + pz' + qz = Ae^{i\omega t}$, then the real and imaginary parts of z are solutions of $x'' + px' + qx = A\cos\omega t$ and $y'' + py' + qy = A\sin\omega t$, respectively.

10.

11. Let $z = Ae^{i2t}$. Then

$$z' = 2iAe^{i2t}$$
$$z'' = (2i)^2 Ae^{i2t},$$

and $z'' + 9z = e^{i2t}$ leads to

$$(2i)^2 Ae^{i2t} + 9Ae^{i2t} = e^{i2t}$$
$$((2i)^2 + 9)A = 1$$
$$A = \frac{1}{5}.$$

Thus, $z = (1/5)e^{i2t}$ is a particular solution of $z'' + 9z = e^{i2t}$ and the imaginary part of this solution, $y = (1/5)\sin 2t$, is the solution of $y'' + 9y = \sin 2t$.

12.

13. Let $z = Ae^{i3t}$. Then

$$z' = (3i)Ae^{i3t}$$
$$z'' = (3i)^2 Ae^{i3t},$$

and $z'' + 7z' + 10z = -4e^{i3t}$ leads to

$$(3i)^2 Ae^{i3t} + 7(3i)Ae^{i3t} + 10Ae^{i3t} = -4e^{i3t}$$
$$((3i)^2 + 7(3i) + 10)A = -4$$
$$A = -\frac{4}{1 + 21i}$$
$$A = -\frac{2}{221} + \frac{42}{221}i.$$

Thus,

$$z = \left(-\frac{2}{221} + \frac{42}{221}i\right)e^{i3t}$$
$$z = \left(-\frac{2}{221} + \frac{42}{221}i\right)(\cos 3t + i\sin 3t)$$
$$z = \left(-\frac{2}{221}\cos 3t - \frac{42}{221}\sin 3t\right) + i\left(\frac{42}{221}\cos 3t - \frac{2}{221}\sin 3t\right)$$

is a solution of $z'' + 7z' + 10z = -4e^{i3t}$. The imaginary part,

$$y = \frac{42}{221}\cos 3t - \frac{2}{221}\sin 3t,$$

is a solution of $y'' + 7y' + 10y = -4\sin 3t$.

14.

15. Let $y = at + b$. Then, $y' = a$, $y'' = 0$, and $y'' + 6y' + 8y = 2t - 3$ leads to $8at + (6a + 8b) = 2t - 3$. Thus, $8a = 2$ and $6a + 8b = -3$, which gives $a = 1/4$ and $b = -9/16$. Therefore, the particular solution is $y = (1/4)t - 9/16$.

16.

17. Let $y = at^3 + bt^2 + ct + d$. Then, substituting

$$\begin{aligned} y' &= 3at^2 + 2bt + c \\ y'' &= 6at + 2b \end{aligned}$$

into $y'' + 3y' + 4y = t^3$, arranging terms,

$$4at^3 + (9a + 4b)t^2 + (6a + 6b + 4c)t + (2b + 3c + 4d) = t^3,$$

and comparing coefficients,

$$\begin{aligned} 4a &= 1 \\ 9a + 4b &= 0 \\ 6a + 6b + 4c &= 0 \\ 2b + 3c + 4d &= 0, \end{aligned}$$

provides $a = 1/4$, $b = -9/16$, $c = 15/32$, and $d = -9/128$. Thus, a particular solution is

$$y = \frac{1}{4}t^3 - \frac{9}{16}t^2 + \frac{15}{32}t - \frac{9}{128}.$$

18.

19. The homogeneous equation $y'' - 4y' - 5y = 0$ has characteristic equation $\lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1) = 0$ with zeros $\lambda_1 = 5$ and $\lambda_2 = -1$. This leads to the homogeneous solution

$$y_h = C_1 e^{5t} + C_2 e^{-t}.$$

The particular solution $y_p = Ae^{-2t}$ has derivatives $y_p' = -2Ae^{-2t}$ and $y_p'' = 4Ae^{-2t}$, which when substituted into the equation $y'' - 4y' - 5y = 4e^{-2t}$ provides

$$\begin{aligned} 4Ae^{-2t} + 8Ae^{-2t} - 5Ae^{-2t} &= 4e^{-2t} \\ 7A &= 4 \\ A &= \frac{4}{7}. \end{aligned}$$

Thus, a particular solution is $y_p = (4/7)e^{-2t}$. This leads to the general solution

$$y = C_1 e^{5t} + C_2 e^{-t} + \frac{4}{7}e^{-2t}.$$

The initial condition $y(0) = 0$ gives

$$0 = C_1 + C_2 + \frac{4}{7}.$$

Differentiating,

$$y' = 5C_1 e^{5t} - C_2 e^{-t} - \frac{8}{7}e^{-2t}.$$

The initial condition $y'(0) = -1$ provides

$$-1 = 5C_1 - C_2 - \frac{8}{7}.$$

This system has solution $C_1 = -1/14$ and $C_2 = -1/2$, leading to the solution

$$y = -\frac{1}{14}e^{5t} - \frac{1}{2}e^{-t} + \frac{4}{7}e^{-2t}.$$

20.

21. The homogeneous equation $y'' - 2y' + 5y = 0$ has characteristic equation $\lambda^2 - 2\lambda + 5 = 0$ with zeros $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$. This leads to the homogeneous solution

$$y_h = e^t(C_1 \cos 2t + C_2 \sin 2t).$$

The particular solution $z = Ae^{it}$ has derivatives

$$\begin{aligned} z' &= iAe^{it} \\ z'' &= i^2 e^{it}, \end{aligned}$$

which, when inserted in the complex equation $z'' - 2z' + 5z = 3e^{it}$, gives

$$\begin{aligned} i^2Ae^{it} - 2iAe^{it} + 5Ae^{it} &= 3e^{it} \\ (i^2 - 2i + 5)A &= 3 \\ A &= \frac{3}{4 - 2i} \\ A &= \frac{3}{5} + \frac{3}{10}i. \end{aligned}$$

This gives the particular solution

$$\begin{aligned} z &= \left(\frac{3}{5} + \frac{3}{10}i\right)e^{it} \\ z &= \left(\frac{3}{5}\cos t + \frac{3}{10}i\right)(\cos t + i \sin t). \end{aligned}$$

The real part of this solution is a particular solution of $y'' - 2y' + 5y = 3 \cos t$.

$$y_p = \frac{3}{5}\cos t - \frac{3}{10}\sin t$$

Thus, the general solution is

$$y = e^t(C_1 \cos 2t + C_2 \sin 2t) + \frac{3}{5}\cos t - \frac{3}{10}\sin t.$$

The initial condition $y(0) = 0$ gives $0 = C_1 + 3/5$. Differentiating,

$$y' = e^t(-2C_1 \sin 2t + 2C_2 \cos 2t) + e^t(C_1 \cos 2t + C_2 \sin 2t) - \frac{3}{5}\sin t - \frac{3}{10}\cos t.$$

The initial condition $y'(0) = -2$ provides $-2 = 2C_2 + C_1 - 3/10$. The solution of this system is $C_1 = -3/5$ and $C_2 = -11/20$. Therefore, the solution is

$$y = e^t(-\frac{3}{5}\cos 2t - \frac{11}{20}\sin 2t) + \frac{3}{5}\cos t - \frac{3}{10}\sin t.$$

22.

23. The homogeneous equation $y'' - 2y' + y = 0$ has characteristic equation $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$, with repeated zero $\lambda = 1$. Thus, the homogeneous solution is

$$y_h = (C_1 + C_2 t)e^t.$$

The particular solution $y_p = at^3 + bt^2 + ct + d$ has derivatives

$$y_p' = 3at^2 + 2bt + c$$
$$y'' = 6at + 2b,$$

which when substituted in $y'' - 2y' + y = t^3$, rearranging, yields

$$at^3 + (-6a + b)t^2 + (6a - 4b + c)t + (2b - 2c + d) = t^3.$$

Thus,

$$\begin{aligned} a &= 1 \\ -6a + b &= 0 \\ 6a - 4b + c &= 0 \\ 2b - 2c + d &= 0, \end{aligned}$$

which has solution $a = 1$, $b = 6$, $c = 18$, and $d = 24$. Thus, the general solution is

$$y = (C_1 + C_2 t)e^t + t^3 + 6t^2 + 18t + 24.$$

The initial condition $y(0) = 1$ gives $1 = C_1 + 24$. Differentiating,

$$y' = C_2 e^t + (C_1 + C_2 t)e^t + 3t^2 + 12t + 18.$$

The initial condition $y'(0) = 0$ gives $0 = C_2 + C_1 + 18$. The system has solution $C_1 = -23$ and $C_2 = 5$. Therefore, the solution is

$$y = (-23 + 5t)e^t + t^3 + 6t^2 + 18t + 24.$$

24.

25. The homogeneous equation $y'' - y' - 2y = 0$ has characteristic equation $\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0$ with zeros $\lambda_1 = 2$ and $\lambda_2 = -1$. Thus, the homogeneous solution is

$$y_h = C_1 e^{2t} + C_2 e^{-t}.$$

The forcing term of $y'' - y' - 2y = 2e^{-t}$ is also a solution of the homogeneous equation, so we multiply by a factor of t and try $y_p = Ate^{-t}$, which has derivatives

$$y' = Ae^{-t}(1 - t)$$
$$y'' = Ae^{-t}(t - 2).$$

Substituting these in $y'' - y' - 2y = 2e^{-t}$ gives

$$\begin{aligned} Ae^{-t}(t - 2) - Ae^{-t}(1 - t) - 2Ate^{-t} &= 2e^{-t} \\ A(t - 2 - 1 + t - 2t) &= 2 \\ -3A &= 2 \\ A &= -\frac{2}{3}. \end{aligned}$$

Thus, a particular solution is $y_p = -(2/3)te^{-t}$.

26.

27. The homogeneous equation $z'' + 9z = 0$ has characteristic equation $\lambda^2 + 9 = 0$, which has zeros $\lambda_1 = 3i$ and $\lambda_2 = -3i$. Thus, the homogeneous solution is

$$z_h = C_1 e^{3it} + C_2 e^{-3it}.$$

The forcing term of $z'' + 9z = e^{3it}$ is also a solution of the homogeneous equation, so multiply by a factor of t and try the particular solution $z_p = Ate^{3it}$. The particular solution has derivatives

$$\begin{aligned} z_p' &= Ae^{3it}(1 + 3it) \\ z_p'' &= 3iAe^{3it}(2 + 3it). \end{aligned}$$

If these are substituted into $z'' + 9z = e^{3it}$, then

$$\begin{aligned} 3iAe^{3it}(2 + 3it) + 9Ate^{3it} &= e^{3it} \\ 3iA(2 + 3it) + 9At &= 1 \\ 6iA &= 1 \\ A &= -\frac{1}{6}i. \end{aligned}$$

Thus,

$$\begin{aligned} z_p &= -\frac{1}{6}ite^{3it} \\ &= -\frac{1}{6}it(\cos 3t + i \sin 3t) \\ &= \frac{1}{6}t \sin 3t + i\left(-\frac{1}{6}t \cos 3t\right). \end{aligned}$$

The imaginary part of this solution is a particular solution of $y'' + 9y = \sin 3t$.

$$y_p = -\frac{1}{6}t \cos 3t$$

28.

29. The homogeneous equation $y'' + 6y' + 9y = 0$ has characteristic equation $\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0$ and repeated zero $\lambda = -3$. Thus, the homogeneous solution is

$$y_h = (C_1 + C_2 t)e^{-3t}.$$

The forcing term of $y'' + 6y' + 9y = 5e^{-3t}$ is a solution of the homogeneous equation, as is te^{-3t}. Consequently, try $y_p = At^2e^{-3t}$, which has derivatives

$$\begin{aligned} y_p' &= -3At^2e^{-3t} + 2Ate^{-3t} \\ y_p'' &= 9At^2e^{-3t} - 12Ate^{-3t} + 2Ae^{-3t}. \end{aligned}$$

Substituting these into $y'' + 6y' + 9y = 5e^{-3t}$ yields

$$\begin{aligned} 2Ae^{-3t} &= 5e^{-3t} \\ 2A &= 5 \\ A &= \frac{5}{2}. \end{aligned}$$

Therefore, $y_p = (5/2)t^2e^{-3t}$ is a solution of $y'' + 6y' + 9y = 5e^{-3t}$.

30.

31. It is easy to see that $y = 1$ is a solution of $y'' + 2y' + 2y = 2$. Next, $z_p = Ae^{i2t}$ has derivatives

$$\begin{aligned} z_p' &= 2iAe^{i2t} \\ z_p'' &= (2i)^2Ae^{i2t}. \end{aligned}$$

Substituting these into $z'' + 2z' + 2z = e^{i2t}$,

$$\begin{aligned} (2i)^2Ae^{i2t} + 2(2i)Ae^{i2t} + 2Ae^{i2t} &= e^{i2t} \\ ((2i)^2 + 2(2i) + 2)A &= 1 \\ A &= \frac{1}{-2+4i} \\ A &= -\frac{1}{10} - \frac{1}{5}i. \end{aligned}$$

Thus,

$$\begin{aligned} z_p &= \left(-\frac{1}{10} - \frac{1}{5}i\right)e^{i2t} \\ z_p &= \left(-\frac{1}{10} - \frac{1}{5}i\right)(\cos 2t + i\sin 2t) \\ z_p &= \left(-\frac{1}{10}\cos 2t + \frac{1}{5}\sin 2t\right) + i\left(-\frac{1}{5}\cos 2t - \frac{1}{10}\sin 2t\right). \end{aligned}$$

The real part of this, $y = -(1/10)\cos 2t + (1/5)\sin 2t$, is a solution of $y'' + 2y' + 2y = \cos 2t$. Thus,

$$y = 1 - \frac{1}{10}\cos 2t + \frac{1}{5}\sin 2t$$

is a solution of $y'' + 2y' + 2y = 2 + \cos 2t$.

32.

33. First, let $y = at + b$, with derivatives $y' = a$ and $y'' = 0$, be a solution of $y'' + 25y = 2 + 3t$. Substituting,

$$25at + 25b = 2 + 3t,$$

which, upon equating coefficients, leads to $a = 2/25$ and $b = 3/25$ and the particular solution $y = (2/25)t + 3/25$. Next, the homogeneous equation $z'' + 25z = 0$ has characteristic $\lambda^2 + 25 = 0$ and zeros $\lambda = \pm 5i$, giving the complex solution

$$z = C_1e^{5it} + C_2e^{-5it}.$$

Because $z = e^{5it}$ is a solution of the homogeneous equation, we try $z = Ate^{5it}$ as a solution of $z'' + 25z = e^{5it}$. The derivatives

$$z' = (5i)Ate^{5it} + Ae^{5it}$$
$$z'' = (5i)^2 Ate^{5it} + 2(5i)Ae^{5it},$$

when substituted in $z'' + 25z = e^{5it}$, provide

$$(5i)^2 Ate^{5it} + 2(5i)Ae^{5it} + 25Ate^{5it} = e^{5it}$$
$$10iA = 1$$
$$A = -\frac{1}{10}i.$$

Thus,

$$z = -\frac{1}{10}ite^{5it} = -\frac{1}{10}it(\cos 5t + i\sin 5t).$$

The real part of this solution, $y = (1/10)t\sin 5t$, is a solution of $y'' + 25y = 2 + 3t + \cos 5t$. Thus,

$$y = (2/25)t + 3/25 + (1/10)t\sin 5t$$

is a solution of $y'' + 25y = \cos 5t$.

34.

35. Substitute $z = Ae^{i2t}$ and its derivatives

$$z' = (2i)Ae^{i2t}$$
$$z'' = (2i)^2 Ae^{i2t}$$

into $z'' + 4z' + 3z = e^{i2t}$.

$$(2i)^2 Ae^{i2t} + 4(2i)Ae^{i2t} + 3Ae^{i2t} = e^{i2t}$$
$$((2i)^2 + 4(2i) + 3)A = 1$$
$$A = -\frac{1}{65} - \frac{8}{65}i.$$

Thus,

$$\begin{aligned} z &= \left(-\frac{1}{65} - \frac{8}{65}i\right)e^{i2t} \\ &= \left(-\frac{1}{65} - \frac{8}{65}i\right)(\cos 2t + i\sin 2t) \\ &= \left(-\frac{1}{65}\cos 2t + \frac{8}{65}\sin 2t\right) + i\left(-\frac{8}{65}\cos 2t - \frac{1}{65}\sin 2t\right). \end{aligned}$$

Thus, $y = -(1/65)\cos 2t + (8/65)\sin 2t$ is a solution of $y'' + 4y' + 3y = \cos 2t$ and $y = -(8/65)\cos 2t - (1/65)\sin 2t$ is a solution of $y'' + 4y' + 3y = \sin 2t$. This means that

$$\begin{aligned} y &= \left(-\frac{1}{65}\cos 2t + \frac{8}{65}\sin 2t\right) + 3\left(-\frac{8}{65}\cos 2t - \frac{1}{65}\sin 2t\right) \\ &= -\frac{5}{13}\cos 2t + \frac{1}{13}\sin 2t \end{aligned}$$

is a solution of $y'' + 4y' + 3y = \cos 2t + 3\sin 2t$.

36.

37. The homogeneous equation $y'' + 4y' + 4y = 0$ has characteristic equation $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0$ with repeated zero $\lambda = -2$ and solution $y = (C_1 + C_2 t)e^{-2t}$. Thus, both e^{-2t} and te^{-2t} are solutions of the homogeneous equation. Thus, try $y = At^2e^{-2t}$ as a particular solution of $y''+4y'+4y = e^{-2t}$.

$$\begin{aligned} y' &= -2At^2e^{-2t} + 2Ate^{-2t} \\ y'' &= 4At^2e^{-2t} - 8Ate^{-2t} + 2Ae^{-2t} \end{aligned}$$

Substitute y and its derivatives in $y'' + 4y' + 4y = e^{-2t}$ to get

$$\begin{aligned} 2Ae^{-2t} &= e^{-2t} \\ 2A &= 1 \\ A &= \frac{1}{2}. \end{aligned}$$

Thus, $y = (1/2)t^2e^{-2t}$ is a solution of $y''+4y'+4y = e^{-2t}$. Next, let $z = Ae^{i2t}$ and

$$\begin{aligned} z' &= (2i)Ae^{i2t} \\ z'' &= (2i)^2Ae^{i2t}. \end{aligned}$$

Substituting in $z'' + 4z' + 4z = e^{i2t}$,

$$\begin{aligned} (2i)^2Ae^{i2t} + 4(2i)Ae^{i2t} + 4Ae^{i2t} &= e^{i2t} \\ ((2i)^2 + 4(2i) + 4)A &= 1 \\ A &= -\frac{1}{8}i, \end{aligned}$$

so

$$z = -\frac{1}{8}ie^{i2t} = -\frac{1}{8}i(\cos 2t + i\sin 2t).$$

The imaginary part of this solution, $y = -(1/8)\cos 2t$, is a solution of $y'' + 4y' + 4y = \sin 2t$. Thus,

$$y = \frac{1}{2}t^2e^{-2t} - \frac{1}{8}\cos 2t$$

is a solution of $y'' + 4y' + 4y = e^{-2t} + \sin 2t$.

38.

39. If $y = (at + b)e^{-4t}$, then

$$\begin{aligned} y' &= e^{-4t}(a - 4at - 4b) \\ y'' &= e^{-4t}(-8a + 16at + 16b), \end{aligned}$$

and substituting in $y'' + 3y' + 2y = te^{-4t}$ gives

$$\begin{aligned} (-8a + 16at + 16b) + 3(a - 4at - 4b) + 2(at + b) &= t \\ 6at + (-5a + 6b) &= t. \end{aligned}$$

Comparing coefficients, $6a = 1$ and $-5a + 6b = 0$. This gives $a = 1/6$ and $b = 5/36$ and $y = ((1/6)t + 5/36)e^{-4t}$ as a solution of $y'' + 3y' + 2y = te^{-4t}$.

40.

41. Let $y = (at^2 + bt + c)e^{-2t}$. Then,

$$\begin{aligned} y' &= e^{-2t}(-2at^2 + (2a - 2b)t + (b - 2c)) \\ y'' &= e^{-2t}(4at^2 + (-8a + 4b)t + (2a - 4b + 4c)). \end{aligned}$$

Substituting in $y'' + 2y' + y = t^2e^{-2t}$,

$$\begin{aligned} (4at^2 + (-8a + 4b)t + (2a - 4b + 4c)) \\ +2(-2at^2 + (2a - 2b)t + (b - 2c)) + (at^2 + bt + c) &= t^2 \\ at^2 + (-4a + b)t + (2a - 2b + c) &= t^2. \end{aligned}$$

Comparing coefficients,

$$\begin{aligned} a &= 1 \\ -4a + b &= 0 \\ 2a - 2b + c &= 0 \end{aligned}$$

and $a = 1$, $b = 4$, and $c = 6$. Thus, $y = (t^2 + 4t + 6)e^{-2t}$ is a solution of $y'' + 2y' + y = t^2e^{-2t}$.

42.

43. The homogeneous equation $y'' + 3y' + 2y = 0$ has characteristic equation $\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$ and solution

$$y = C_1e^{-t} + C_2e^{-2t}.$$

Thus, instead of $y = (at^2 + bt + c)e^{-2t}$, we will try $y = t(at^2 + bt + c)e^{-2t}$. After some work, the derivatives are found.

$$\begin{aligned} y' &= e^{-2t}(-2at^3 + (3a - 2b)t^2 + (2b - 2c)t + c) \\ y'' &= e^{-2t}(4at^3 + (-12a + 4b)t^2 + (6a - 8b + 4c)t + (2b - 4c)) \end{aligned}$$

Substitute these into the differential equation $y'' + 3y' + 2y = t^2e^{-2t}$. After some computation,

$$-3at^2 + (6a - 2b)t + (2b - c) = t^2.$$

Comparing coefficients,

$$\begin{aligned} -3a &= 1 \\ 6a - 2b &= 0 \\ 2b - c &= 0 \end{aligned}$$

leading to $a = -1/3$, $b = -1$, and $c = -2$. Thus,

$$y = t\left(-\frac{1}{3}t^2 - t - 2\right)e^{-2t}$$

is a solution of $y'' + 3y' + 2y = t^2e^{-2t}$.

44.

45. If $z(t) = x(t) + iy(t)$ is a solution of $z'' + pz' + qz = Ae^{(a+bi)t}$, then

$$(x'' + iy'') + p(x' + iy') + q(x + iy) = Ae^{at}e^{ibt}$$
$$(x'' + px' + qx) + i(y'' + py' + qy) = Ae^{at}\cos bt + iAe^{at}\sin bt.$$

Equating real and imaginary parts,

$$x'' + px' + qx = Ae^{at}\cos bt$$
$$y'' + py' + qy = Ae^{at}\sin bt.$$

46.

47. If $z(t) = x(t) = iy(t)$ is a solution of $z'' + z' + z = te^{it}$, then

$$(x'' + iy'') + (x' + iy') + (x + iy) = t(\cos t + i\sin t)$$
$$(x'' + x' + x) + i(y'' + y' + y) = t\cos t + it\sin t.$$

Comparing imaginary parts,

$$y'' + y' + y = t\sin t.$$

Thus, if $z = (at + b)e^{it}$ and its derivatives,

$$z' = e^{it}(a + i(at + b))$$
$$z'' = e^{it}(-at - b + 2ai)$$

are substituted into $z'' + z' + z = te^{it}$, then

$$(-at - b + 2ai) + (a + i(at + b)) + (at + b) = t$$
$$iat + ((1 + 2i)a + ib) = t.$$

Thus,

$$ia = 1$$
$$(1 + 2i)a + ib = 0,$$

and $a = -i$ and $b = 1 + 2i$. Thus, a particular solution of $z'' + z' + z = te^{it}$ is

$$\begin{aligned} z &= (-it + (1 + 2i))e^{it} \\ &= (1 + i(2 - t))(\cos t + i\sin t) \\ &= (\cos t - (2 - t)\sin t) + i((2 - t)\cos t + \sin t). \end{aligned}$$

Therefore, the imaginary part, $y = (2 - t)\cos t + \sin t$, is a solution of $y'' + y' + y = t\sin t$.

Section 4.6

1. The homogeneous equation $y'' + 9y = 0$ has $y_1(t) = \cos 3t$ and $y_2(t) = \sin 3t$ as a fundamental set of solutions. The Wronskian is

$$\begin{aligned} W(\cos 3t, \sin 3t) &= \begin{vmatrix} \cos 3t & \sin 3t \\ -3\sin 3t & 3\cos 3t \end{vmatrix} \\ &= 3\cos^2 3t + 3\sin^2 3t \\ &= 3. \end{aligned}$$

Form the solution

$$y_p = v_1 y_1 + v_2 y_2$$

where v_1 and v_2 are to be determined. Indeed,

$$\begin{aligned} v_1' &= \frac{-y_2 g(t)}{W(y_1, y_2)} \\ &= \frac{-\sin 3t \tan 3t}{3} \\ &= -\frac{1}{3}\frac{\sin^2 3t}{\cos 3t} \\ &= -\frac{1}{3}\frac{1-\cos^2 3t}{\cos 3t} \\ &= -\frac{1}{3}(\sec 3t - \cos 3t). \end{aligned}$$

Thus,

$$v_1 = -\frac{1}{9}\ln|\sec 3t + \tan 3t| + \frac{1}{9}\sin 3t.$$

Secondly,

$$\begin{aligned} v_2' &= \frac{y_1 g(t)}{W(y_1, y_2)} \\ &= \frac{\cos 3t \tan 3t}{3} \\ &= \frac{1}{3}\sin 3t. \end{aligned}$$

Thus,

$$v_2 = -\frac{1}{9}\cos 3t.$$

Inserting these results in $y_p = v_1 y_1 + v_2 y_2$,

$$\begin{aligned} y_p &= \left[-\frac{1}{9}\ln|\sec 3t + \tan 3t| + \frac{1}{9}\sin 3t\right]\cos 3t + \left(-\frac{1}{9}\cos 3t\right)\sin 3t \\ &= -\frac{1}{9}\cos 3t \ln|\sec 3t + \tan 3t| \end{aligned}$$

is the particular solution we seek.

2. A fundamental system of solutions to the homogeneous equation is $y_1(t) = \cos 2t$ and $y_2(t) = \sin 2t$. We look for a solution of the form

$$y = v_1 y_1 + v_2 y_2 = v_1 \cos 2t + v_2 \sin 2t.$$

Differentiating we get

$$y' = v_1' \cos 2t + v_2' \sin 2t - 2v_1 \cos 2t + 2v_2 \sin 2t.$$

To simplify future calculations we set

$$v_1' \cos 2t + v_2' \sin 2t = 0,$$

so $y' = -2v_1 \cos 2t + 2v_2 \sin 2t$, and

$$y'' = -2v_1' \cos 2t + 2v_2' \sin 2t - 4[v_1 \cos 2t + v_2 \sin 2t].$$

Adding, we get

$$\begin{aligned} y'' + 4y &= -2v_1' \cos 2t + 2v_2' \sin 2t \\ &= \sec 2t. \end{aligned}$$

We must solve the system

$$\begin{aligned} v_1' \cos 2t + v_2' \sin 2t &= 0, \\ -2v_1' \cos 2t + 2v_2' \sin 2t &= \sec 2t. \end{aligned}$$

The solutions are

$$v_1' = -\frac{1}{2} \tan 2t \quad \text{and} \quad v_2' = \frac{1}{2}.$$

Integrating we get

$$v_1(t) = \frac{1}{4} \ln(\cos 2t) \quad \text{and} \quad v_2(t) = \frac{t}{2}.$$

Thus the solution is

$$\begin{aligned} y(t) &= v_1 \cos 2t + v_2 \sin 2t \\ &= \frac{1}{4} \cos 2t \cdot \ln(\cos 2t) + \frac{t}{2} \sin 2t. \end{aligned}$$

3. The homogeneous equation $y'' - y = 0$ has $y_1(t) = e^t$ and $y_2(t) = e^{-t}$ as a fundamental set of solutions. The Wronskian is

$$\begin{aligned} W(e^t, e^{-t}) &= \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} \\ &= -2. \end{aligned}$$

Form the solution

$$y_p = v_1 y_1 + v_2 y_2$$

where v_1 and v_2 are to be determined. Indeed,

$$\begin{aligned} v_1' &= \frac{-y_2 g(t)}{W(y_1, y_2)} \\ &= \frac{-e^{-t}(t+3)}{-2} \\ &= \frac{1}{2} e^{-t}(t+3). \end{aligned}$$

Integrating by parts,

$$v_1 = -\frac{1}{2} e^{-t}(t+3) - \frac{1}{2} e^{-t}.$$

Secondly,

$$\begin{aligned} v_2' &= \frac{y_1 g(t)}{W(y_1, y_2)} \\ &= \frac{e^t(t+3)}{-2} \\ &= -\frac{1}{2} e^t(t+3). \end{aligned}$$

Integrating by parts,

$$v_2 = -\frac{1}{2}e^t(t+3) + \frac{1}{2}e^t.$$

Inserting these results in $y_p = v_1 y_1 + v_2 y_2$,

$$\begin{aligned} y_p &= \left(-\frac{1}{2}e^{-t}(t+3) - \frac{1}{2}e^{-t}\right)e^t + \left(-\frac{1}{2}e^t(t+3) + \frac{1}{2}e^t\right)e^{-t} \\ &= -\frac{1}{2}(t+3) - \frac{1}{2} - \frac{1}{2}(t+3) + \frac{1}{2} \\ &= -(t+3) \end{aligned}$$

is the particular solution we seek.

4. A fundamental system of solutions to the homogeneous equation is $x_1(t) = e^{-t}$ and $x_2(t) = e^{3t}$. We look for a solution of the form

$$x(t) = v_1(t)x_1(t) + v_2(t)x_2(t) = v_1(t)e^{-t} + v_2(t)e^{3t}.$$

Differentiating we get

$$x' = v_1'e^{-t} + v_2'e^{3t} - v_1e^{-t} + 3v_2e^{3t}.$$

To simplify future calculations we set

$$v_1'e^{-t} + v_2'e^{3t} = 0.$$

Then $x' = -v_1e^{-t} + 3v_2e^{3t}$, and

$$x'' = -v_1'e^{-t} + 3v_2'e^{3t} + v_1e^{-t} + 9v_2e^{3t}.$$

Adding, we get

$$\begin{aligned} x'' - 2x' - 3x &= -v_1'e^{-t} + 3v_2'e^{3t} \\ &= 4e^{3t}. \end{aligned}$$

Thus we must solve the system

$$\begin{aligned} v_1'e^{-t} + v_2'e^{3t} &= 0 \\ -v_1'e^{-t} + 3v_2'e^{3t} &= 4e^{3t}. \end{aligned}$$

The solutions are

$$v_1' = -e^{4t} \quad \text{and} \quad v_2' = 1.$$

Integrating we get

$$v_1(t) = -\frac{1}{4}e^{4t} \quad \text{and} \quad v_2(t) = t.$$

Thus the solution is

$$x(t) = -\frac{1}{4}e^{3t} + te^{3t}.$$

5. The homogeneous equation $y'' - 2y' + y = 0$ has $y_1(t) = e^t$ and $y_2(t) = te^t$ as a fundamental set of solutions. The Wronskian is

$$\begin{aligned} W(e^t, te^t) &= \begin{vmatrix} e^t & te^t \\ e^t & te^t + e^t \end{vmatrix} \\ &= e^{2t}. \end{aligned}$$

Form the solution

$$y_p = v_1 y_1 + v_2 y_2$$

where v_1 and v_2 are to be determined. Indeed,

$$\begin{aligned} v_1' &= \frac{-y_2 g(t)}{W(y_1, y_2)} \\ &= \frac{(-te^t)e^t}{e^{2t}} \\ &= -t. \end{aligned}$$

Thus,

$$v_1 = -\frac{1}{2}t^2.$$

Secondly,

$$\begin{aligned} v_2' &= \frac{y_1 g(t)}{W(y_1, y_2)} \\ &= \frac{e^t e^t}{e^{2t}} \\ &= 1 \end{aligned}$$

Thus,

$$v_2 = t.$$

Inserting these results in $y_p = v_1 y_1 + v_2 y_2$,

$$\begin{aligned} y_p &= -\frac{1}{2}t^2 e^t + t(te^t) \\ &= -\frac{1}{2}t^2 e^t + t^2 e^t \\ &= \frac{1}{2}t^2 e^t. \end{aligned}$$

is the particular solution we seek.

6. A fundamental system of solutions to the homogeneous equation is $x_1(t) = e^{2t}$ and $x_2(t) = te^{2t}$. We look for a solution of the form

$$x(t) = v_1(t)x_1(t) + v_2(t)x_2(t) = v_1(t)e^{2t} + v_2(t)te^{2t}.$$

Differentiating we get

$$x' = v_1' e^{2t} + v_2' te^{2t} + 2v_1 e^{2t} + v_2(1 + 2t)e^{2t}.$$

To simplify future calculations we set

$$v_1' e^{2t} + v_2' te^{2t} = 0.$$

Then $x' = 2v_1 e^{2t} + v_2(1 + 2t)e^{2t}$, and

$$x'' = 2v_1' e^{2t} + v_2'(1 + 2t)e^{2t} + 4v_1 e^{2t} + v_2(4 + 4t)e^{2t}.$$

Adding, we get

$$\begin{aligned} x'' - 4x' + 4x &= 2v_1' e^{2t} + v_2'(1 + 2t)e^{2t} \\ &= e^{2t}. \end{aligned}$$

Thus we must solve the system

$$v_1' e^{2t} + v_2' t e^{2t} = 0,$$
$$2v_1' e^{2t} + v_2'(1+2t)e^{2t} = e^{2t}.$$

The solutions are

$$v_1' = -t \quad \text{and} \quad v_2' = 1.$$

Integrating we get

$$v_1(t) = -t^2/2 \quad \text{and} \quad v_2(t) = t.$$

Thus the solution is

$$x(t) = \frac{1}{2}t^2 e^{2t}.$$

7. The homogeneous equation $x'' + x = 0$ has $x_1(t) = \cos t$ and $x_2(t) = \sin t$ as a fundamental set of solutions. The Wronskian is

$$\begin{aligned} W(\cos t, \sin t) &= \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} \\ &= 1. \end{aligned}$$

Form the solution

$$x_p = v_1 x_1 + v_2 x_2$$

where v_1 and v_2 are to be determined. Indeed,

$$\begin{aligned} v_1' &= \frac{-\sin t \tan^2 t}{1} \\ &= -\sin t(\sec^2 t - 1) \\ &= -\sec t \tan t + \sin t. \end{aligned}$$

Thus,

$$v_1 = -\sec t - \cos t.$$

Secondly,

$$\begin{aligned} v_2' &= \frac{\cos t \tan^2 t}{1} \\ &= \cos t(\sec^2 t - 1) \\ &= \sec t - \cos t. \end{aligned}$$

Thus,

$$v_2 = \ln|\sec t + \tan t| - \sin t.$$

Inserting these results in $x_p = v_1 x_1 + v_2 x_2$,

$$\begin{aligned} x_p &= (-\sec t - \cos t)\cos t + (\ln|\sec t + \tan t| - \sin t)\sin t \\ &= -1 - \cos^2 t + \sin t \ln|\sec t + \tan t| - \sin^2 t \\ &= -2 + \sin t \ln|\sec t + \tan t| \end{aligned}$$

is the particular solution we seek.

8. A fundamental system of solutions to the homogeneous equation is $x_1(t) = \cos t$ and $x_2(t) = \sin t$. We look for a solution of the form

$$x(t) = v_1(t)x_1(t) + v_2(t)x_2(t) = v_1(t)\cos t + v_2(t)\sin t.$$

Differentiating we get

$$x' = v_1' \cos t + v_2' \sin t - v_1 \sin t + v_2 \cos t.$$

To simplify future calculations we set

$$v_1' \cos t + v_2' \sin t = 0.$$

Then $x' = -v_1 \sin t + v_2 \cos t$, and

$$x'' = -v_1' \sin t + v_2' \cos t - v_1 \cos t - v_2 \sin t.$$

Adding, we get

$$\begin{aligned} x'' + x &= -v_1' \sin t + v_2' \cos t \\ &= \sec^2 t. \end{aligned}$$

Thus we must solve the system

$$\begin{aligned} v_1' \cos t + v_2' \sin t &= 0, \\ -v_1' \sin t + v_2' \cos t &= \sec^2 t. \end{aligned}$$

The solutions are

$$v_1' = -\sin t \sec^2 t \quad \text{and} \quad v_2' = \sec t.$$

Integrating we get

$$v_1(t) = -\sec t \quad \text{and} \quad v_2(t) = \ln(\sec t + \tan t).$$

Thus the solution is

$$\begin{aligned} x(t) &= v_1(t)\cos t + v_2(t)\sin t \\ &= -\sec t \cos t + \sin t \cdot \ln(\sec t + \tan t) \\ &= -1 + \sin t \cdot \ln(\sec t + \tan t). \end{aligned}$$

9. A fundamental system of solutions to the homogeneous equation is $x_1(t) = \cos t$ and $x_2(t) = \sin t$. We look for a solution of the form

$$x(t) = v_1(t)x_1(t) + v_2(t)x_2(t) = v_1(t)\cos t + v_2(t)\sin t.$$

Differentiating we get

$$x' = v_1' \cos t + v_2' \sin t - v_1 \sin t + v_2 \cos t.$$

To simplify future calculations we set

$$v_1' \cos t + v_2' \sin t = 0.$$

Then $x' = -v_1 \sin t + v_2 \cos t$, and

$$x'' = -v_1' \sin t + v_2' \cos t - v_1 \cos t - v_2 \sin t.$$

Adding, we get

$$\begin{aligned} x'' + x &= -v_1' \sin t + v_2' \cos t \\ &= \sin^2 t. \end{aligned}$$

Thus we must solve the system

$$\begin{aligned} v_1' \cos t + v_2' \sin t &= 0, \\ -v_1' \sin t + v_2' \cos t &= \sin^2 t. \end{aligned}$$

The solutions are

$$v_1' = -\sin^3 t \quad \text{and} \quad v_2' = \sin^2 t \cos t.$$

Integrating we get

$$\begin{aligned} v_1(t) &= -\int \sin^3 t \, dt \\ &= \int (\cos^2 t - 1) \sin t \, dt \\ &= -\frac{1}{3} \cos^3 t + \cos t, \end{aligned}$$

and

$$v_2(t) = \int \sin^2 t \cos t \, dt = \frac{1}{3} \sin^3 t.$$

Thus the solution is

$$\begin{aligned} x(t) &= v_1(t) \cos t + v_2(t) \sin t \\ &= -\frac{1}{3} \cos^4 t + \cos^2 t + \frac{1}{3} \sin^4 t \\ &= \frac{1}{3}(\sin^2 t - \cos^2 t) + \cos^2 t \\ &= \frac{1}{3} \sin^2 t + \frac{2}{3} \cos^2 t \\ &= \frac{1}{3}[1 + \cos^2 t]. \end{aligned}$$

10. A fundamental set of solutions to the homogeneous equation are $y_1(t) = e^{-t}$ and $y_2(t) = te^{-t}$. We will look for a solution of the form

$$y = v_1 y_1 + v_2 y_2 = v_1 e^{-t} + v_2 t e^{-t}.$$

Differentiating we get

$$y' = v_1' e^{-t} + v_2' t e^{-t} - v_1 e^{-t} + v_2 (1 - t) e^{-t}.$$

To simplify future calculations we set

$$v_1' e^{-t} + v_2' t e^{-t} = 0.$$

Then $y' = -v_1 e^{-t} + v_2(1 - t)e^{-t}$, and

$$y'' = -v_1' e^{-t} + v_2'(1 - t) e^{-t} + v_1 e^{-t} + v_2 (2 - t) e^{-t}.$$

Adding, we get

$$\begin{aligned} y'' + 2y' + y &= -v_1' e^{-t} + v_2'(1 - t) e^{-t} \\ &= t^5 e^{-t}. \end{aligned}$$

After cancelling the common term e^{-t}, we see that we must solve the system

$$\begin{aligned} v_1' + t v_2' &= 0 \\ -v_1' + (1-t)v_2' &= t^5. \end{aligned}$$

The solutions are

$$v_1' = -t^6 \quad \text{and} \quad v_2' = t^5.$$

Integrating we get

$$v_1(t) = -\frac{1}{7}t^7 \quad \text{and} \quad v_2(t) = \frac{1}{6}t^6.$$

Thus the solution is

$$\begin{aligned} y(t) &= v_1 e^{-t} + v_2 t e^{-t} \\ &= -\frac{1}{7}t^7 e^{-t} + \frac{1}{6}t^6 t e^{-t} \\ &= \frac{1}{42}t^7 e^{-t}. \end{aligned}$$

11. The homogeneous equation $y'' + y = 0$ has $y_1(t) = \cos t$ and $y_2(t) = \sin t$ as a fundamental set of solutions. The Wronskian is

$$\begin{aligned} W(\cos t, \sin t) &= \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} \\ &= 1. \end{aligned}$$

Form the solution

$$y_p = v_1 y_1 + v_2 y_2$$

where v_1 and v_2 are to be determined. Indeed,

$$\begin{aligned} v_1' &= \frac{-\sin t(\tan t + \sin t + 1)}{1} \\ &= -\frac{\sin^2 t}{\cos t} - \sin^2 t - \sin t \\ &= \frac{\cos^2 t - 1}{\cos t} - \frac{1 - \cos 2t}{2} - \sin t \\ &= \cos t - \sec t - \frac{1}{2} + \frac{1}{2}\cos 2t - \sin t. \end{aligned}$$

Thus,

$$v_1 = \sin t - \ln|\sec t + \tan t| - \frac{1}{2}t + \frac{1}{4}\sin 2t + \cos t.$$

Secondly,

$$\begin{aligned} v_2' &= \frac{\cos t(\tan t + \sin t + 1)}{1} \\ &= \sin t + \cos t \sin t + \cos t \\ &= \sin t + \frac{1}{2}\sin 2t + \cos t. \end{aligned}$$

Thus,

$$v_2 = -\cos t - \frac{1}{4}\cos 2t + \sin t.$$

Inserting these results in $y_p = v_1 y_1 + v_2 y_2$,

$$\begin{aligned} y_p &= \left(\sin t - \ln|\sec t + \tan t| - \frac{1}{2}t + \frac{1}{4}\sin 2t + \cos t\right)\cos t \\ &\quad + \left(-\cos t - \frac{1}{4}\cos 2t + \sin t\right)\sin t \\ &= 1 - \frac{1}{2}t\cos t - \cos t \ln|\sec t + \tan t| + \frac{1}{4}(\sin 2t\cos t - \sin t\cos 2t) \\ &= 1 - \frac{1}{2}t\cos t - \cos t \ln|\sec t + \tan t| + \frac{1}{4}\sin(2t - t) \\ &= 1 - \frac{1}{2}t\cos t - \cos t \ln|\sec t + \tan t| + \frac{1}{4}\sin t \end{aligned}$$

is the particular solution we seek.

12. A fundamental system of solutions to the homogeneous equation is $y_1(t) = \cos t$ and $y_2(t) = \sin t$. We look for a solution of the form

$$y(t) = v_1(t)y_1(t) + v_2(t)y_2(t) = v_1(t)\cos t + v_2(t)\sin t.$$

Differentiating we get

$$y' = v_1'\cos t + v_2'\sin t - v_1\sin t + v_2\cos t.$$

To simplify future calculations we set

$$v_1'\cos t + v_2'\sin t = 0.$$

Then $y' = -v_1\sin t + v_2\cos t$, and

$$y'' = -v_1'\sin t + v_2'\cos t - v_1\cos t - v_2\sin t.$$

Adding, we get

$$\begin{aligned} y'' + y &= -v_1'\sin t + v_2'\cos t \\ &= \sec t + \cos t - 1. \end{aligned}$$

Thus we must solve the system

$$\begin{aligned} v_1'\cos t + v_2'\sin t &= 0, \\ -v_1'\sin t + v_2'\cos t &= \sec t + \cos t - 1. \end{aligned}$$

The solutions are

$$\begin{aligned} v_1' &= -\tan t - \sin t\cos t + \sin t \quad \text{and} \\ v_2' &= 1 + \cos^2 t - \cos t. \end{aligned}$$

Integrating we get

$$\begin{aligned} v_1(t) &= \ln(\cos t) = \frac{1}{2}\cos^2 t - \cos t \quad \text{and} \\ v_2(t) &= \frac{3}{2}t + \frac{1}{2}\sin t\cos t - \sin t. \end{aligned}$$

Thus the solution is

$$\begin{aligned} y(t) &= v_1(t)\cos t + v_2(t)\sin t \\ &= \cos t \cdot \ln(\cos t) + \frac{1}{2}\cos t + \frac{3}{2}\sin t - 1. \end{aligned}$$

13. The formulae given in the text depend upon the fact that the coefficient of y'' is 1. We start by dividing our equation by t^2.

$$y'' + \frac{3}{t}y' - \frac{3}{t^2}y = \frac{1}{t^3}.$$

First, check $y_1(t) = t$ is a solution.

$$y'' + \frac{3}{t}y' - \frac{3}{t^2}y = (0) + \frac{3}{t}(1) - \frac{3}{t^2}(t) = 0.$$

Check that $y_2(t) = t^{-3}$ is a solution.

$$\begin{aligned} y'' + \frac{3}{t}y' - \frac{3}{t^2}y &= (12t^{-5}) + \frac{3}{t}(-3t^{-4}) - \frac{3}{t^2}(t^{-3}) \\ &= 12t^{-5} - 9t^{-5} - 3t^{-5} \\ &= 0. \end{aligned}$$

Calculate the Wronskian.

$$W(t, t^{-3}) = \begin{vmatrix} t & t^{-3} \\ 1 & -3t^{-4} \end{vmatrix} = -4t^{-3}.$$

Next,

$$\begin{aligned} v1' &= \frac{-y2g(t)}{W} \\ &= \frac{-t^{-3}t^{-3}}{-4t^{-3}} \\ &= \frac{1}{4}t^{-3}. \end{aligned}$$

Thus,

$$v_1 = -\frac{1}{8}t^{-2}.$$

Next,

$$\begin{aligned} v_2' &= \frac{y_1 g(t)}{W} \\ &= \frac{tt^{-3}}{-4t^{-3}} \\ &= -\frac{1}{4}t. \end{aligned}$$

Thus,

$$v_2 = -\frac{1}{8}t^2.$$

Form

$$\begin{aligned} y_p &= v_1 y_1 + v_2 y_2 \\ &= \left(-\frac{1}{8}t^{-2}\right)t + \left(-\frac{1}{8}t^2\right)t^{-3} \\ &= -\frac{1}{4t}. \end{aligned}$$

Thus, the general solution is

$$y(t) = C_1 t + \frac{C_2}{t^3} - \frac{1}{4t}.$$

14.

Section 4.7

1. (a) If $x_p = a\cos\omega t$, then

$$\begin{aligned} x_p' &= -a\omega \sin\omega t \\ x_p'' &= -a\omega^2 \cos\omega t. \end{aligned}$$

Substituting these in $x'' + \omega_0^2 x = A\cos\omega t$,

$$\begin{aligned} -a\omega^2 \cos\omega t + \omega_0^2 a \cos\omega t &= A\cos\omega t \\ a(\omega_0^2 - \omega^2)\cos\omega t &= A\cos\omega t. \end{aligned}$$

Comparing coefficients, $a = A/(\omega_0^2 - \omega^2)$ and

$$x_p = \frac{A}{\omega_0^2 - \omega^2}\cos\omega t.$$

(b) Substitute $z = ae^{i\omega t}$ into $z'' + \omega_0^2 z = Ae^{i\omega t}$ to get

$$\begin{aligned} [(i\omega)^2 - \omega_0^2]\, ae^{i\omega t} &= Ae^{i\omega t} \\ (\omega_0^2 - \omega^2)z &= Ae^{i\omega t} \\ z &= \frac{A}{\omega_0^2 - \omega^2}\, e^{i\omega t}. \end{aligned}$$

The real part of this is the solution we seek.

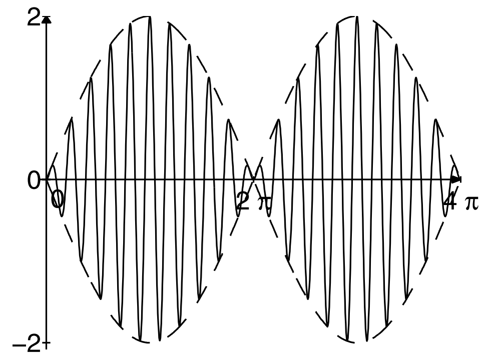
$$x_p = \frac{A}{\omega_0^2 - \omega^2}\cos\omega t$$

2.

3. The mean frequency is $\overline{\omega} = 21/2$ and the half difference is $\delta = 1/2$. Thus,

$$\begin{aligned} \cos 10t - \cos 11t &= \cos\left(\frac{21}{2} - \frac{1}{2}\right)t - \cos\left(\frac{21}{2} + \frac{1}{2}\right)t, \\ &= 2\sin\frac{1}{2}t \sin\frac{21}{2}t. \end{aligned}$$

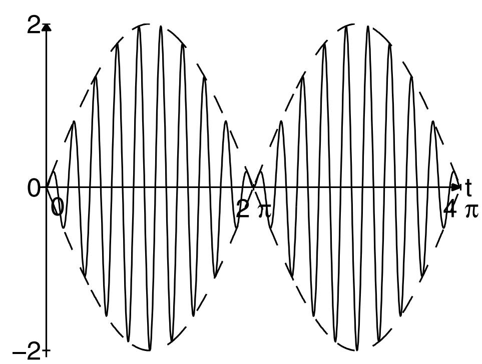
The envelope is $y(t) = \pm 2\sin(1/2)t$. The graph of the envelope is dashed in the following figure.



4. The mean frequency is $\overline{\omega} = 19/2$ and the half difference is $\delta = 1/2$. Thus,

$$\begin{aligned}\cos 9t - \cos 10t &= \cos\left(\frac{19}{2} - \frac{1}{2}\right)t - \cos\left(\frac{19}{2} + \frac{1}{2}\right)t,\\ &= 2\sin\frac{1}{2}t\sin\frac{19}{2}t.\end{aligned}$$

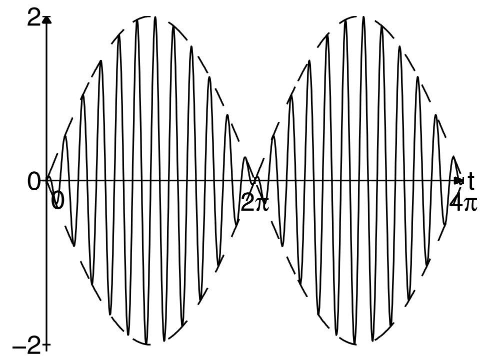
The envelope is $y(t) = \pm 2\sin(1/2)t$. The graph of the envelope is dashed in the following figure.



5. The mean frequency is $\overline{\omega} = 23/2$ and the half difference is $\delta = 1/2$. Thus,

$$\begin{aligned}\sin 12t - \sin 11t &= \sin\left(\frac{23}{2} - \frac{1}{2}\right)t - \sin\left(\frac{23}{2} + \frac{1}{2}\right)t,\\ &= 2\sin\frac{1}{2}t\cos\frac{23}{2}t.\end{aligned}$$

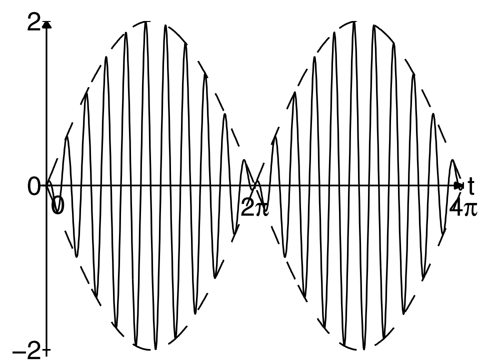
The envelope is $y(t) = \pm 2\sin(1/2)t$. The graph of the envelope is dashed in the following figure.



6. The mean frequency is $\overline{\omega} = 21/2$ and the half difference is $\delta = 1/2$. Thus,

$$\begin{aligned}\sin 11t - \sin 10t &= \sin\left(\frac{21}{2} - \frac{1}{2}\right)t - \sin\left(\frac{21}{2} + \frac{1}{2}\right)t,\\ &= 2\sin\frac{1}{2}t\cos\frac{21}{2}t.\end{aligned}$$

The envelope is $y(t) = \pm 2\sin(1/2)t$. The graph of the envelope is dashed in the following figure.



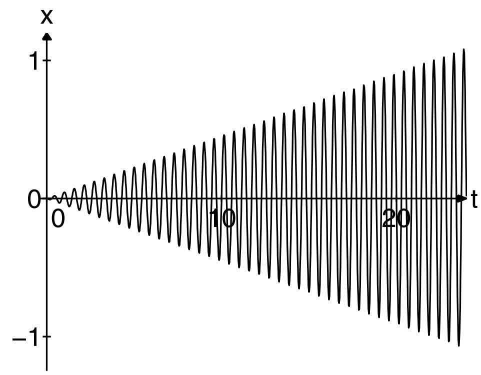
7. If we set $\overline{\omega} = (\omega_0 + \omega)/2$, and $\delta = (\omega_0 - \omega)/2$, we get

$$\begin{aligned}x(t) &= \frac{\sin\delta t}{2\overline{\omega}\delta}\sin\overline{\omega}t\\ &= \frac{\sin\delta t}{\delta t}\frac{t\sin\overline{\omega}t}{2\overline{\omega}}.\end{aligned}$$

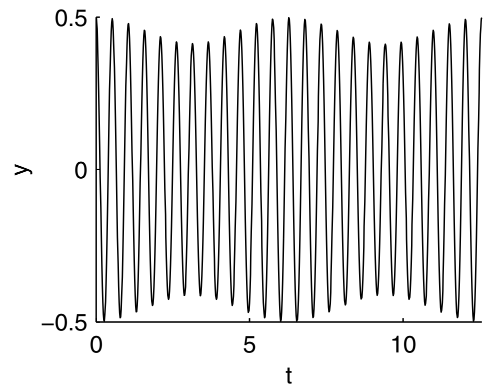
As $\omega \to \omega_0$, we have $\delta \to 0$ and $\sin\delta t/\delta t \to 1$. In addition $\overline{\omega} \to \omega_0$, so

$$x(t) \to \frac{t\sin\omega_0 t}{2\omega_0},$$

the resonance solution. The graph of x with $\omega = 10.99$ is shown in the following figure.



8. The "slow" and "fast" frequencies are present in each case, although the slow frequency becomes less pronounced as y_0 increases. Below is the plot for $y_0 = 0.5$.



9. (a) The displacement satisfies the differential equation $x'' + 4x = 4\cos\omega t$, with initial conditions $x(0) = x'(0) = 0$. The solution is

$$x(t) = \frac{4}{4-\omega^2}(\cos\omega t - \cos 2t).$$

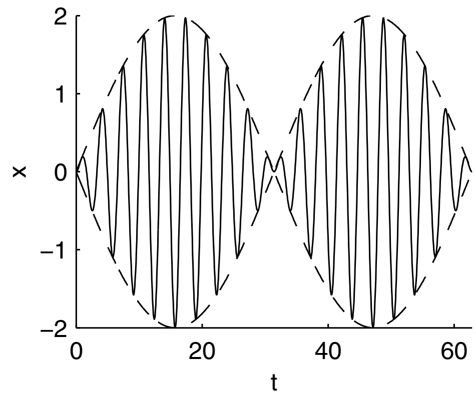
(b) If we set $\overline{\omega} = (2+\omega)/2$, and $\delta = (2-\omega)/2$, the solution becomes

$$x(t) = \frac{2}{\overline{\omega}\delta}\sin\delta t \sin\overline{\omega} t.$$

We will take $\omega = 1.8$, which is near to $\omega_0 = 2$. Then $\overline{\omega} = 1.9$, and $\delta = 0.1$, so

$$x(t) = \frac{2}{0.19}\sin 0.1t \sin 1.9t.$$

The graph of x and its envelope is presented in the following figure.



10. (a) As in the text we find the particular solution

$$x_p(t) = \frac{A}{2\omega_0} t \sin \omega_0 t = \frac{2}{5} t \sin 5t.$$

The solution of the homogeneous equation is $x_h(t) = C_1 \cos 5t + C_2 \sin 5t$, so our solution has the form

$$x(t) = C_1 \cos 5t + C_2 \sin 5t + \frac{2}{5} t \sin 5t.$$

Our initial conditions are

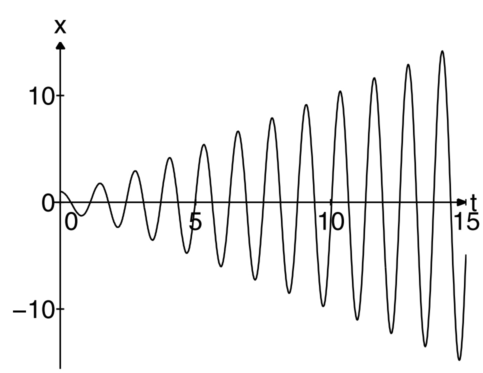
$$1 = x(0) = C_1$$
$$0 = x'(0) = 5C_2,$$

so the solution is

$$x(t) = \cos 5t + \frac{2}{5} t \sin 5t.$$

The particular solution $x_p(t)$ has a factor of t so its amplitude will grow, indicating a resonant solution.

(b)



11. (a) The equation for the current is $I'' + 4I = -12\omega \sin \omega t$. We find a particular solution of the form $I_p(t) = A \cos \omega t + B \sin \omega t$. Solving for A and B we find that

$$I_p(t) = \frac{12\omega}{\omega^2 - 4} \sin \omega t.$$

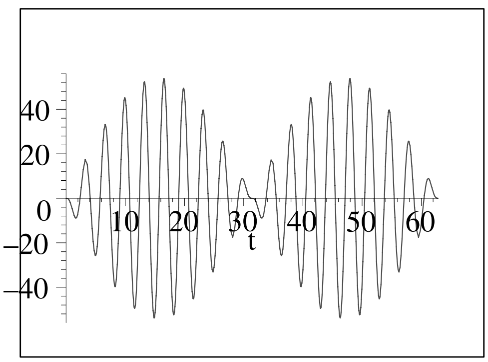
Since the general solution of the homogeneous equation is $I_h(t) = C_1 \cos 2t + C_2 \sin 2t$, we know that the current has the form

$$I(t) = C_1 \cos 2t + C_2 \sin 2t + \frac{12\omega}{\omega^2 - 4} \sin \omega t.$$

The initial conditions $I(0) = 0$, and $I'(0) = 0$ imply that $C_1 = 0$, and $C_2 = -12\omega/(\omega^2 - 4)$. Hence the solution is

$$I(t) = \frac{6\omega}{\omega^2 - 4}(2 \sin \omega t - \omega \sin 2t).$$

With $\omega = 1.8$ we get the following graph of the solution.



(b) The equation for the current is $I'' + 4I = -24 \sin 2t$. We find a particular solution of the form $I_p(t) = t[A \cos 2t + B \sin 2t]$. Solving for A and B we find that

$$I_p(t) = 6t \cos 2t.$$

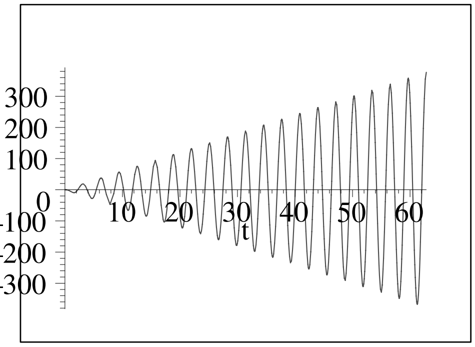
The general solution is

$$I(t) = 6t \cos 2t + C_1 \cos 2t + C_2 \sin 2t.$$

The initial conditions $I(0) = 0$, and $I'(0) = 0$ imply that $C_1 = 0$, and $C_2 = -3$. Hence the solution is

$$I(t) = 6t \cos 2t - 3 \sin 2t.$$

It is plotted in the following figure.



12. The characteristic polynomial is $P(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2 = \lambda^2 + \lambda + 4$. Hence $2c = 1$ and $\omega_0^2 = 4$. Hence

$$R(\omega) = \sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2} = \sqrt{(\omega^2 - 4)^2 + \omega^2}$$
$$\phi(\omega) = \operatorname{arccot}\left(\frac{\omega_0^2 - \omega^2}{2c\omega}\right) = \operatorname{arccot}\left(\frac{4 - \omega^2}{\omega}\right).$$

For the steady-state solution to $x'' + x' + 4x = 3\cos 2t$ we use $\omega = 2 = \omega_0$. Hence $R(2) = 2$ and $\phi = \operatorname{arccot} 0 = \pi/2$. The solution is

$$x(t) = A\frac{1}{R}\cos(\omega t - \phi) = 1.5\cos(2t - \pi/2) = 1.5\sin 2t.$$

13. The characteristic polynomial is $P(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2 = \lambda^2 + 2\lambda + 2$. Hence $2c = 2$ and $\omega_0^2 = 2$. Therefore

$$R(\omega) = \sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2} = \sqrt{(\omega^2 - 2)^2 + 4\omega^2}$$
$$\phi(\omega) = \operatorname{arccot}\left(\frac{\omega_0^2 - \omega^2}{2c\omega}\right) = \operatorname{arccot}\left(\frac{2 - \omega^2}{2\omega}\right).$$

For the steady-state solution to $x'' + 2x' + 2x = 3\sin 4t$, we look first for a complex solution to $z'' + 2z' + 2z = 3e^{4it}$. The solution will be $x(t) = \operatorname{Im} z(t)$. The complex solution is

$$z(t) = H(i\omega)\cdot 3e^{4it} = \frac{1}{R}\cdot 3e^{4it - i\phi}.$$

The real solution is then

$$x(t) = \operatorname{Im} z(t) = \frac{3}{R}\sin(4t - \phi).$$

We use $\omega = 4$. Hence $R(4) = \sqrt{260} \approx 15.8114$ and $\phi(4) = \operatorname{arccot}(-7/4) \approx 2.6224$. The solution is

$$x(t) = \frac{3}{\sqrt{260}}\sin(4t - \phi) = \frac{3}{\sqrt{260}}\cos(4t - \phi - \pi/2).$$

14. The characteristic polynomial is $P(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2 = \lambda^2 + 2\lambda + 4$. Hence$2c = 2$ and $\omega_0^2 = 4$, and

$$R(\omega) = \sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2} = \sqrt{(\omega^2 - 4)^2 + 4\omega^2}$$
$$\phi(\omega) = \operatorname{arccot}\left(\frac{\omega_0^2 - \omega^2}{2c\omega}\right) = \operatorname{arccot}\left(\frac{4 - \omega^2}{2\omega}\right).$$

For the steady-state solution to $x'' + 2x' + 4x = 2\sin 2\pi t$, we look first for a complex solution to $z'' + 2z' + 4z = 2e^{2\pi i t}$. The solution will be $x(t) = \operatorname{Im} z(t)$. The complex solution is

$$z(t) = H(i\omega) \cdot 2e^{2\pi i t} = \frac{1}{R} \cdot 2e^{2\pi i t - i\phi}.$$

The real solution is then

$$x(t) = \operatorname{Im} z(t) = \frac{2}{R}\sin(2\pi t - \phi).$$

We use $\omega = 2\pi$. Hence $R(2\pi) = 37.6382$ and $\phi = 2.8012$. The solution is

$$x(t) = 0.0531\sin(2\pi t - 2.8012).$$

15. We want to find the complex solution of $z'' + 4z' + 8z = 3e^{i2\pi t}$. Note that the frequency of the forcing term is $\omega = 2\pi$. The equation has characteristic polynomial $P(\lambda) = \lambda^2 + 4\lambda + 9$, so

$$P(i\omega) = (i\omega)^2 + 4(i\omega) + 8 = (8 - \omega^2) + 4i\omega,$$

which has magnitude and phase defined by

$$R(\omega) = \sqrt{(8 - \omega^2)^2 + 16\omega^2}$$
$$\cot\phi(\omega) = \frac{8 - \omega^2}{4\omega}.$$

With $\omega = 2\pi$, $R(2\pi) \approx 40.2808$ and $\phi(2\pi) \approx 2.4678$, leading to the complex solution

$$\begin{aligned} z(t) &= H(i2\pi) \cdot 3e^{i2\pi t} \\ &= \frac{1}{R(2\pi)} e^{-i\phi(2\pi)} \cdot 3e^{i2\pi t} \\ &= \frac{1}{40.2808} e^{-i(2.4678)} \cdot 3e^{i2\pi t} \\ &= 0.0745e^{i(2\pi t - 2.4678)}. \end{aligned}$$

The steady-state solution of $x'' + 4x' + 8x = 3\cos 2\pi t$ is the real part of this solution, namely

$$x(t) = 0.0745\cos(2\pi t - 2.4678).$$

16. We will use the complex method, and look for a solution of the equation $z'' + 5z' + 4z = 2e^{2it}$ of the form $z(t) = ae^{2it}$. Then our particular solution will be $x_p = \operatorname{Im} z$. Differentiating we get

$$z'' + 5z' + 4z = a((2i)^2 + 5 \cdot (2i) + 4)e^{2it} = 10aie^{2it} = 2e^{2it}.$$

Hence $a = -i/5$, $z(t) = -ie^{2it}/5$, and $x_p(t) = \operatorname{Im} z(t) = -\cos 2t/5$.

The characteristic polynomial is $P(\lambda) = \lambda^2 + 5\lambda + 4 = (\lambda + 1)(\lambda + 4)$, which has roots -1 and -4. Hence the general solution to the homogenous equation is $x_h(t) = C_1 e^{-t} + C_2 e^{-4t}$. The general solution to the inhomogeneous equation is

$$x(t) = -\frac{1}{5}\cos 2t + C_1 e^{-t} + C_2 e^{-4t}.$$

The initial conditions imply that

$$\begin{aligned} 1 = x(0) &= -\frac{1}{5} + C_1 + C_2 \\ 0 = x'(0) &= -C_1 - 4C_2 \end{aligned}$$

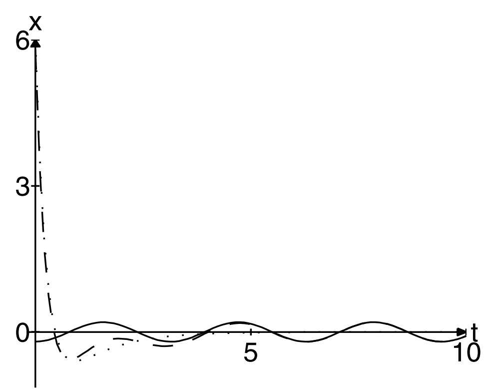
Hence we find C_1 and C_2 which solve the equations

$$\begin{aligned} C_1 + C_2 &= \frac{6}{5}. \\ -C_1 - 4C_2 &= 0 \end{aligned}$$

The solutions are $C_1 = 8/5$ and $C_2 = -2/5$, so the solution to the initial value problem is

$$x(t) = -\frac{1}{5}\cos 2t + \frac{8}{5}e^{-t} - \frac{2}{5}e^{-4t}.$$

The steady-state response is the particular solution $x_p(t) = -\cos 2t/5$, and the transient response is $x_h(t) = [8e^{-t} - 2e^{-4t}]/5$. In the following plot the graph of the solution to the initial value problem is the dashed curve, the transient response is dotted, and the steady-state solution is solid.



17. We will use the complex method, and look for a solution of the equation $z'' + 7z' + 10z = 3e^{3it}$ of the form $z(t) = ae^{3it}$. Then our particular solution will be $x_p = \operatorname{Re} z$. Differentiating we get

$$z'' + 7z' + 10z = a((3i)^2 + 7 \cdot (3i) + 10)e^{3it} = a(1 + 21i)e^{3it} = 3e^{3it}.$$

Hence

$$a = \frac{3}{1 + 21i} = 3\frac{1 - 21i}{442},$$

so

$$\begin{aligned} z(t) &= 3\frac{1-21i}{442}e^{3it} \\ &= \frac{3}{442}[1-21i][\cos 3t + i\sin 3t] \\ &= \frac{3}{442}[(\cos 3t + 21\sin 3t) + i(\sin 3t - 21\cos 3t), \end{aligned}$$

and $x_p(t) = \operatorname{Re} z(t) = 3(\cos 3t + 21\sin 3t)/442$.

The characteristic polynomial is $P(\lambda) = \lambda^2 + 7\lambda + 10 = (\lambda+2)(\lambda+5)$, which has roots -2 and -5. Hence the general solution to the homogenous equation is $x_h(t) = C_1e^{-2t} + C_2e^{-5t}$. The general solution to the inhomogeneous equation is

$$x(t) = \frac{3}{442}(\cos 3t + 21\sin 3t) + C_1e^{-2t} + C_2e^{-5t}.$$

The initial conditions imply that

$$\begin{aligned} -1 = x(0) &= \frac{3}{442} + C_1 + C_2 \\ 0 = x'(0) &= \frac{189}{442} - 2C_1 - 5C_2, \end{aligned}$$

or

$$\begin{aligned} C_1 + C_2 &= -\frac{445}{442} \\ 2C_1 + 5C_2 &= \frac{189}{442} \end{aligned}.$$

The solutions are $C_1 = -71/39$ and $C_2 = 83/102$, so the solution to the initial value problem is

$$x(t) = \frac{3}{442}(\cos 3t + 21\sin 3t) - \frac{71}{39}e^{-2t} + \frac{83}{109}e^{-5t}.$$

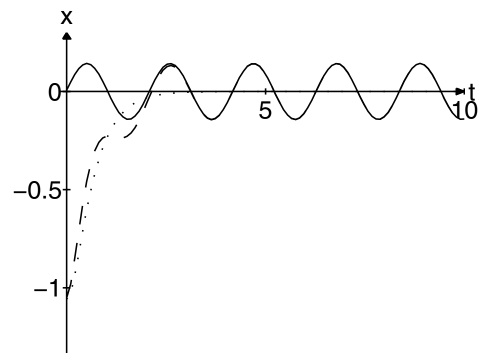
The steady-state solution is the particular solution

$$x_p(t) = \frac{3}{442}(\cos 3t + 21\sin 3t),$$

and the transient response is

$$x_h(t) = -\frac{71}{39}e^{-2t} + \frac{83}{109}e^{-5t}.$$

In the following plot the graph of the solution to the initial value problem is the dashed curve, the transient response is dotted, and the steady-state solution is solid.



18. We will use the complex method, and look for a solution of the equation $z'' + 2z' + 2z = e^{2it}$ of the form $z(t) = ae^{2it}$. Then our particular solution will be $x_p = \operatorname{Re} z$. Differentiating we get

$$z'' + 2z' + 2z = a[(2i)^2 + 2 \cdot (2i) + 2]e^{2it} = a[-2 + 4i]e^{2it} = e^{2it}.$$

Hence

$$a = \frac{1}{-2 + 4i} = \frac{-2 - 4i}{20} = -\frac{1 + 2i}{10},$$

so

$$\begin{aligned} z(t) &= -\frac{1 + 2i}{10} e^{2it} \\ &= -\frac{1}{10}[1 + 2i][\cos 2t + i \sin 2t] \\ &= -\frac{1}{10}[(\cos 2t - 2\sin 2t) + i(2\cos 2t + \sin 2t)], \end{aligned}$$

and $x_p(t) = \operatorname{Re} z(t) = (2\sin 2t - \cos 2t)/10$.

The characteristic polynomial is $P(\lambda) = \lambda^2 + 2\lambda + 2$, which has the complex roots $\lambda = -1 \pm i$. Hence the general solution to the homogeneous equation is $x_h(t) = e^{-t}[C_1 \cos t + C_2 \sin t]$. The general solution to the inhomogeneous equation is

$$x(t) = \frac{1}{10}(2\sin 2t - \cos 2t) + e^{-t}[C_1 \cos t + C_2 \sin t].$$

The initial conditions imply that

$$\begin{aligned} 0 = x(0) &= -\frac{1}{10} + C_1 \\ 2 = x'(0) &= \frac{2}{5} - C_1 + C_2. \end{aligned}$$

We solve these equations, finding $C_1 = 1/10$ and $C_2 = 17/10$, so the solution to the initial value problem is

$$x(t) = \frac{1}{10}(2\sin 2t - \cos 2t) + e^{-t}\left[\frac{1}{10}\cos t + \frac{17}{10}\sin t\right].$$

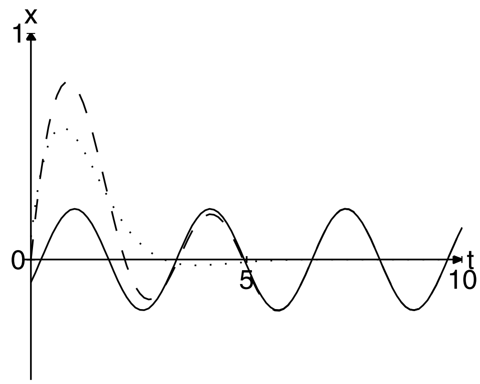
The steady-state solution is the particular solution

$$x_p(t) = (2\sin 2t - \cos 2t)/10,$$

and the transient response is

$$x_h = e^{-t}[\cos t + 17 \sin t]/10.$$

In the following plot the graph of the solution to the initial value problem is the dashed curve, the transient response is dotted, and the steady-state solution is solid.



19. We will use the complex method, and look for a solution of the equation $z'' + 4z' + 5z = 3e^{it}$ of the form $z(t) = ae^{it}$. Then our particular solution will be $x_p = \operatorname{Im} z$. Differentiating we get

$$z'' + 4z' + 5z = a[i^2 + 4i + 5]e^{it} = a[4 + 4i]e^{it} = 3e^{it}.$$

Hence $a = 3/(4 + 4i) = 3(1 - i)/8$, so

$$\begin{aligned} z(t) &= \frac{3}{8}[1 - i]e^{it} \\ &= \frac{3}{8}[1 - i][\cos t + i \sin t] \\ &= \frac{3}{8}[(\cos t + \sin t) + i(\sin t - \cos t)], \end{aligned}$$

and $x_p(t) = \operatorname{Im} z(t) = 3(\sin t - \cos t)/8$.

The characteristic polynomial is $P(\lambda) = \lambda^2 + 4\lambda + 5$, which has the complex roots $\lambda = -2 \pm i$. Hence the general solution to the homogeneous equation is $x_h(t) = e^{-2t}[C_1 \cos t + C_2 \sin t]$. The general solution of the inhomogenous equation is

$$x(t) = \frac{3}{8}(\sin t - \cos t) + e^{-2t}[C_1 \cos t + C_2 \sin t].$$

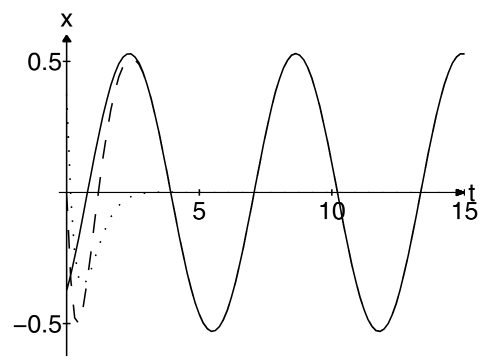
The initial conditions imply

$$\begin{aligned} 0 = x(0) &= -\frac{3}{8} + C_1 \\ -3 = x'(0) &= \frac{3}{8} - 2C_1 + C_2. \end{aligned}$$

We solve these equations, finding $C_1 = 3/8$ and $C_2 = -21/8$, so the solution to the initial value problem is

$$x(t) = \frac{3}{8}(\sin t - \cos t) + \frac{1}{8}e^{-2t}[3\cos t - 21\sin t].$$

The steady-state solution is the particular solution $x_p(t) = 3(\sin t - \cos t)/8$, and the transient response is $x_h = e^{-2t}[3\cos t - 21\sin t]/8$. In the following plot the graph of the solution to the initial value problem is the dashed curve, the transient response is dotted, and the steady-state solution is solid.

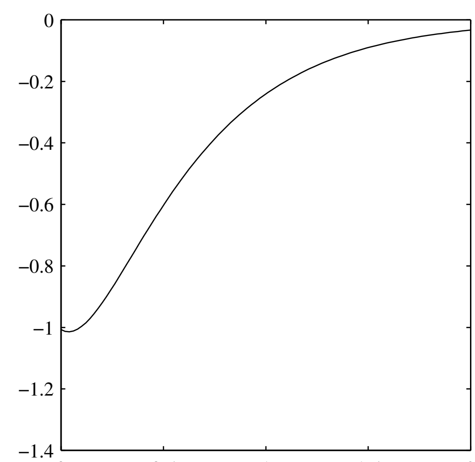


20.

21. In Exercise 17, we found the transient solution

$$x_h(t) = -\frac{71}{39}e^{-2t} + \frac{83}{109}e^{-5t}.$$

Because there are two exponential terms, we will use the more slowing decaying term to determine a time constant. Thus, let $T_c = 1/2$. What follows is the plot of the transient solution on the interval $[0, 4T_c] = [0, 2]$.

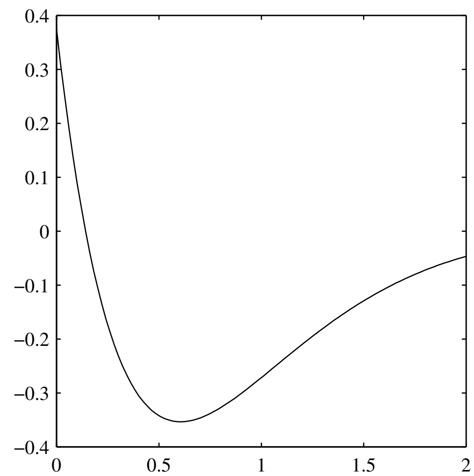


22.

23. In Exercise 19, we found the transient solution

$$x_h = e^{-2t}[3\cos t - 21\sin t]/8.$$

Thus, the time constant is $T_c = 1/2$. What follows is the plot of the transient solution on the interval $[0, 4T_c] = [0, 2]$.



24.

25. We want to find the complex solution of $z'' + 2z' + 5z = 2e^{i3t}$. Note that the frequency of the forcing term is $\omega = 3$. The equation has characteristic polynomial $P(\lambda) = \lambda^2 + 2\lambda + 5$, so

$$P(i\omega) = (i\omega)^2 + 2(i\omega) + 5 = (5 - \omega^2) + 2i\omega,$$

which has magnitude and phase defined by

$$R(\omega) = \sqrt{(5-\omega^2)^2 + 4\omega^2}$$
$$\cot\phi(\omega) = \frac{5-\omega^2}{2\omega}.$$

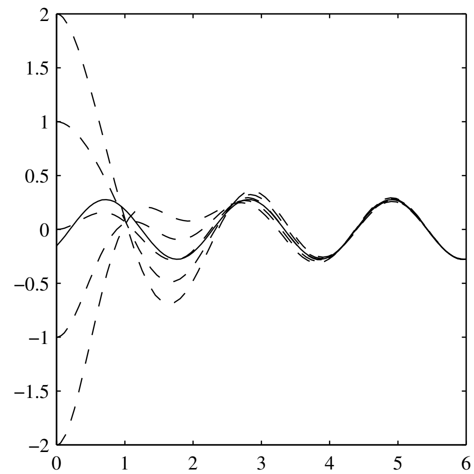
With $\omega = 3$, $R(3) = \sqrt{52} \approx 7.2111$ and $\phi(3) = \operatorname{arccot}(-2/3) \approx 2.1588$, leading to the complex solution

$$z(t) = \frac{1}{\sqrt{52}} e^{-\phi i} \cdot 2e^{3it} = \frac{1}{\sqrt{13}} \epsilon^{i(3t-\phi)}.$$

The steady-state solution of $x'' + 2x' + 5x = 2\cos 3t$ is the real part of this solution, namely

$$x(t) = \frac{1}{\sqrt{13}} \cos(3t - \phi).$$

Note, in the following figure, that the solutions with the initial conditions $(-2, 0)$, $(-1, 0)$, $(0, 0)$, $(1, 0)$, and $(2, 0)$, drawn as dashed lines, all converge to the steady-state solution (solid line).



26.

27. To find the complex solution of $z'' + 0.4z' + 2z = e^{it}$, note that the frequency of the forcing term is $\omega = 1$. The equation has characteristic polynomial $P(\lambda) = \lambda^2 + 0.4\lambda + 2$, so

$$P(i\omega) = (i\omega)^2 + 0.4(i\omega) + 2 = (2 - \omega^2) + 0.4i\omega,$$

which has magnitude and phase defined by

$$\begin{aligned} R(\omega) &= \sqrt{(2-\omega^2)^2 + 0.16\omega^2} \\ \cot\phi(\omega) &= \frac{2-\omega^2}{0.4\omega}. \end{aligned}$$

Thus, the transfer function is

$$H(i\omega) = \frac{1}{P(i\omega)} = \frac{1}{R(\omega)e^{i\phi(\omega)}} = \frac{1}{R(\omega)}e^{-i\phi(\omega)} = G(\omega)e^{-i\phi(\omega)},$$

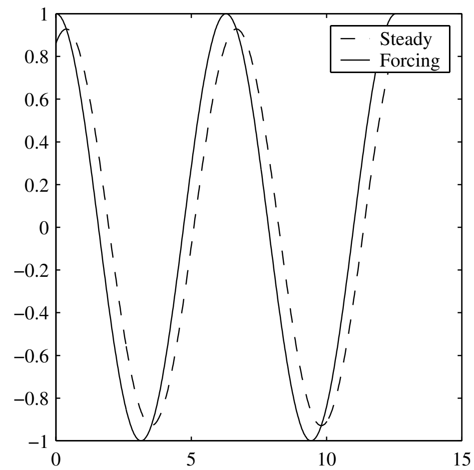
where $G(\omega) = 1/R(\omega)$ is the gain. Thus,

$$z(t) = H(i\omega)e^{it} = G(\omega)e^{-i\phi(\omega)}e^{it} = G(\omega)e^{i(t-\phi(\omega))},$$

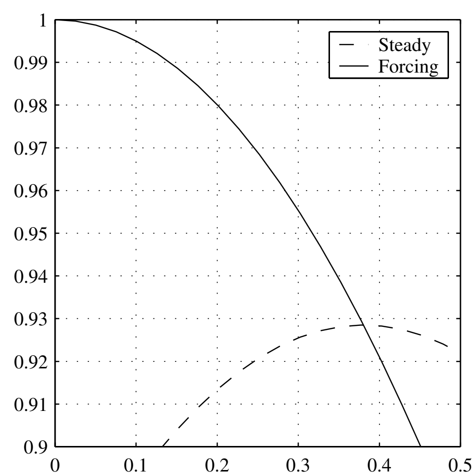
with $\omega = 1$, is the solution of the complex equation, and the real part, $x(t) = G(1)\cos(t - \phi(1))$, is the steady-state solution of $x'' + 0.4x' + 2x = \cos t$. Using the formulae above, $G(1) = 1/\sqrt{1.16} \approx 0.9285$ and $\phi(1) = \text{arccot}(2.5) \approx 0.3805$. Thus, the steady state solution is

$$x(t) = \frac{1}{\sqrt{1.16}}\cos(t - \phi).$$

Note that the amplitude of the steady-state is the gain (not always the case), due to the fact that the amplitude of the forcing function, $\cos t$, is 1. Thus, the gain is easily read in the following graph, where one can estimate the gain by recording the amplitude of the steady-state response (plotted as a dashed line).



Further, note that the steady-state in the figure is shifted to the right of the forcing function. By zooming in on the upper left corner and adding a grid to the figure, one can estimate the phase shift.



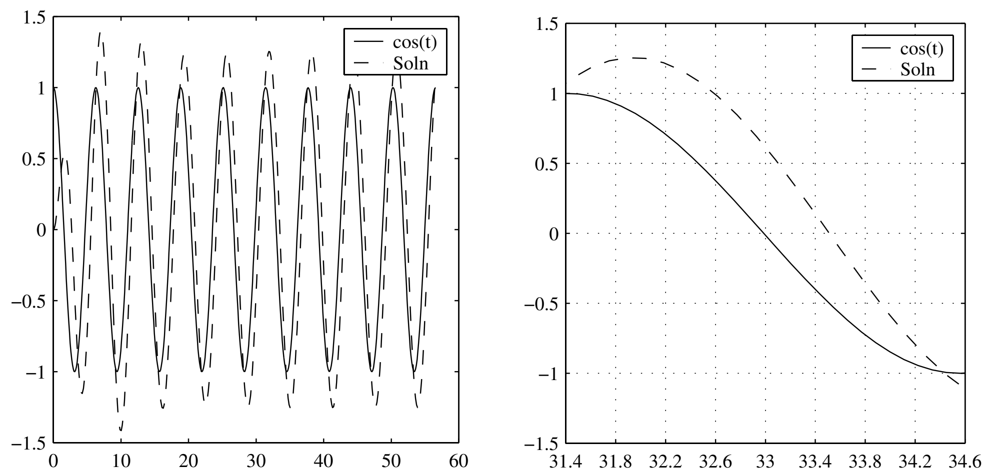
Note that the steady-state response is shifted about 0.38 units to the right of the forcing function. It is important to note that this reading of gain and phase is facilitated by the choice of forcing function. Because the forcing function $\cos t$ has amplitude 1, the gain is easily read as the amplitude of the steady-state response. Similarly, because $\omega = 1$ is $\cos t$, the phase is a simple shift. Things would be more complicated (but still doable) if we used $2\cos 3t$ as the forcing function.

28.

29. The equation $x'' + 0.4x' + 1.69x = \cos t$ has characteristic polynomial $P(\lambda) = \lambda^2 + 0.4\lambda + 1.69$, having roots $\lambda = -(1/5) \pm \sqrt{165}/10$. Consequently, the time constant is

$$T_c = \frac{1}{c} = \frac{1}{1/5} = 5.$$

Thus, to avoid transients, we let at least $4T_c$ pass before examining the solution. Note that in the figure on the left that the solution settles down to its steady-state after about $[0, 4T_c] = [0, 20]$.



In the image on the right, we've zoomed in on the interval $[10\pi, 11\pi]$, which is well beyond [0, 20]. Note that the amplitude of the solution is about 1.25, which is the gain. Further, note that the cosine peaks at about 31.4, but the first peak of the solution occurs at approximately 32.0. Thus, the phase is 32.0-31.4, or 0.6. The careful reader will want to compare these results with the actual gain and phase, 1.2538 and 0.5254.

30.

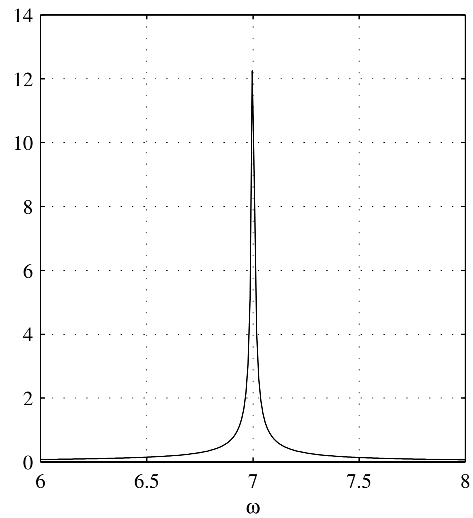
31. Note that the characteristic polynomial of $z'' + 0.01z' + 49z = Ae^{i\omega t}$ is $P(\lambda) = \lambda^2 + 0.01\lambda + 49$. Thus,

$$P(i\omega) = (i\omega)^2 + 0.01(i\omega) + 49 = (49 - \omega^2) + 0.01i\omega.$$

Thus, the gain is

$$G(\omega) = \frac{1}{R(\omega)} = \frac{1}{\sqrt{(49 - \omega^2)^2 + 0.0001\omega^2}}.$$

Plotting the gain on the frequency interval [6, 8], one easily sees that the maximum gain occurs at about $w \approx 7$.



32.

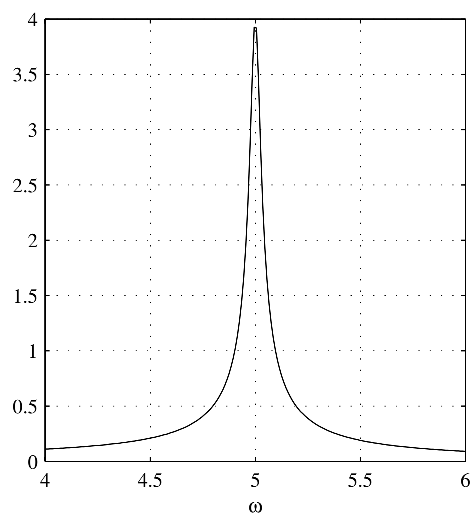
33. Note that the characteristic polynomial of $z'' + 0.05z' + 25z = Ae^{i\omega t}$ is $P(\lambda) = \lambda^2 + 0.05\lambda + 25$. Thus,

$$P(i\omega) = (i\omega)^2 + 0.05(i\omega) + 25 = (25 - \omega^2) + 0.05i\omega.$$

Thus, the gain is

$$G(\omega) = \frac{1}{R(\omega)} = \frac{1}{\sqrt{(25 - \omega^2)^2 + 0.0025\omega^2}}.$$

Plotting the gain on the frequency interval [4, 6], one easily sees that the maximum gain occurs at about $w \approx 5$.



34.

35. The characteristic polynomial of $z'' + 2cz' + \omega_0^2 z = Ae^{i\omega t}$ is $P(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2$. Thus,

$$P(i\omega) = (i\omega)^2 + 2c(i\omega) + \omega_0^2 = (\omega_0^2 - \omega^2) + 2ci\omega.$$

Thus, the gain is

$$G(\omega) = \frac{1}{R(\omega)} = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}}.$$

Taking the derivative,

$$\begin{aligned} G'(\omega) &= -\frac{1}{2}((\omega_0^2 - \omega^2)^2 + 4c^2\omega^2)^{-3/2}(-4\omega(\omega_0^2 - \omega^2) + 8c^2\omega) \\ &= ((\omega_0^2 - \omega^2)^2 + 4c^2\omega^2)^{-3/2}(2\omega(\omega_0^2 - \omega^2) - 4c^2\omega). \end{aligned}$$

Note that the first factor of G is always positive, so critical values are determined by setting the remaining factors equal to zero. Thus,

$$2\omega(\omega_0^2 - \omega^2 - 2c^2) = 0$$

leads to the critical values

$$\omega = 0 \qquad \text{and} \qquad \omega = \sqrt{\omega_0^2 - 2c^2},$$

provided, of course, that $\omega_0^2 > 2c^2$. Practically, we are not interested in a forcing function with zero frequency, so we concentrate on $\omega = \sqrt{\omega_0^2 - 2c^2}$. It is not difficult to show that the derivative of G is positive to the left of this critical value and negative to the right. Thus, we have a maximum at $\omega_{\text{res}} = \sqrt{\omega_0^2 - 2c^2}$, which is the resonant frequency for the driven oscillator. Examining the equation $y'' + 0.01y' + 49y = A\cos\omega t$, the maximum gain occurs at

$$\omega_{\text{res}} = \sqrt{\omega_0^2 - 2c^2} = \sqrt{7^2 - 2(0.005)^2},$$

which, to four decimal places, equals 7.0000, agreeing nicely with the estimate found in Exercise 31.

36.

37. See the derivation of the formula in Exercise 35. Because $y'' + 0.05y' + 25y = A\sin\omega t$, the maximum gain occurs at

$$\omega_{\text{res}} = \sqrt{\omega_0^2 - 2c^2} = \sqrt{5^2 - 2(0.025)^2} \approx 4.9999,$$

which agrees nicely with the result in Exercise 33.

38.

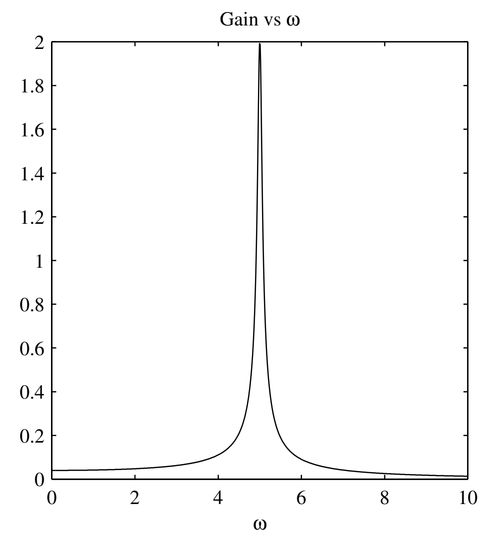
39. The equation $y'' + 0.1y' + 25y = \cos\omega t$ has characteristic polynomial $P(\lambda) = \lambda^2 + 0.1\lambda + 25$. Thus,

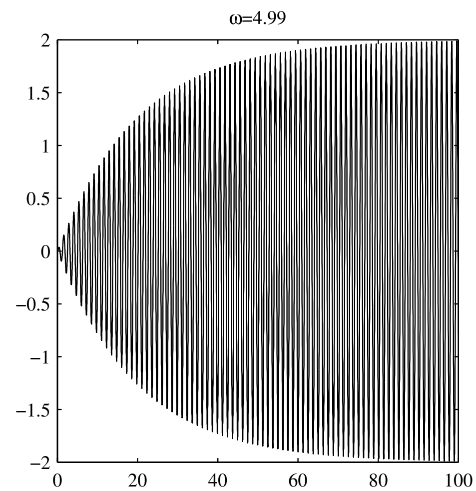
$$P(i\omega) = (i\omega)^2 + 0.1(i\omega) + 25 = (25 - \omega^2) + 0.1i\omega.$$

The gain is

$$G(\omega) = \frac{1}{R(\omega)} = \frac{1}{\sqrt{(25-\omega^2)^2 + 0.01\omega^2}}.$$

At the left, the gain is plotted versus the frequency, indicating a maximum gain near $\omega = 5$.

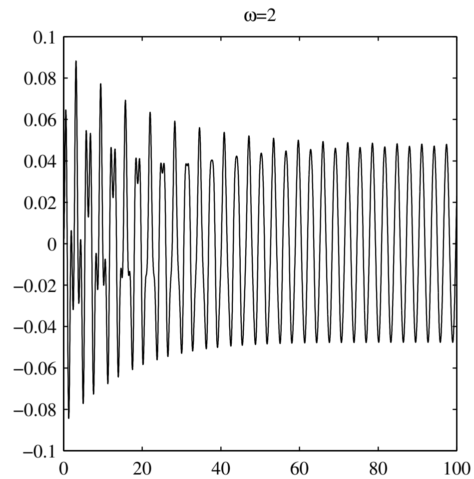




This estimate of the resonant frequency is verified with the calculation

$$\omega_{\text{res}} = \sqrt{\omega_0^2 - 2c^2} = \sqrt{5^2 - 2(0.05)^2} \approx 4.9995.$$

In the figure on the right, careful attention was paid to eliminating transients. In this underdamped case, $T_c = 1/c = 1/0.05 = 20$. Thus, we must go beyond $4T_c = 80$ to eliminate most of the transient behavior. Of course, this extended time interval allows truncation error to propogate, so we set the relative error tolerance of our Matlab solver to 1×10^{-8} to draw the solution in the right figure above, with $\omega = 4.99$ set near the resonant frequency. Note the gain in amplitude is about double that of the forcing function, $\cos 4.99t$. In the figure that follows, $\omega = 2$ was chosen at quite a distance from the resonant frequency.



Note the severe attenuation of the amplitude.

40.

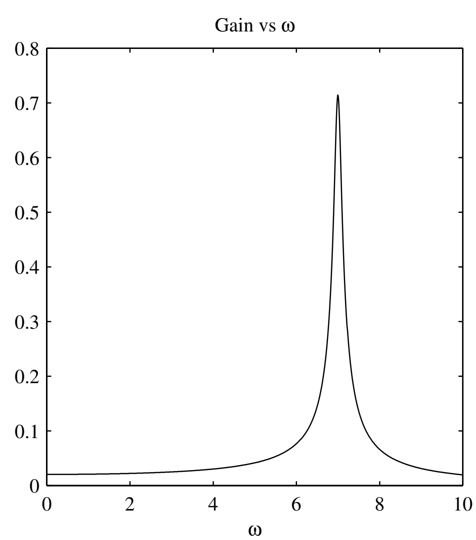
41. The equation $y'' + 0.2y' + 49y = \cos \omega t$ has characteristic polynomial $P(\lambda) = \lambda^2 + 0.2\lambda + 49$. Thus,

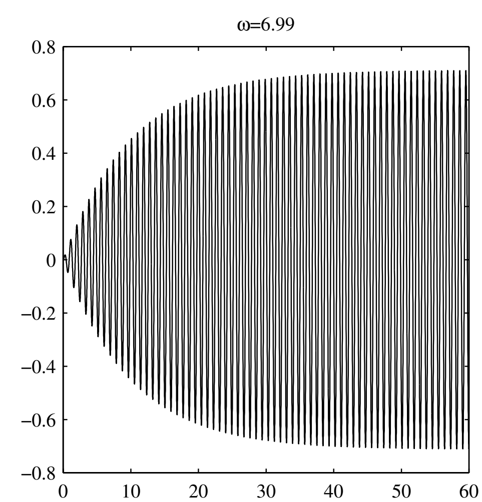
$$P(i\omega) = (i\omega)^2 + 0.2(i\omega) + 49 = (49 - \omega^2) + 0.2i\omega.$$

The gain is

$$G(\omega) = \frac{1}{R(\omega)} = \frac{1}{\sqrt{(49 - \omega^2)^2 + 0.04\omega^2}}.$$

At the left, the gain is plotted versus the frequency, indicating a maximum gain near $\omega = 7$.

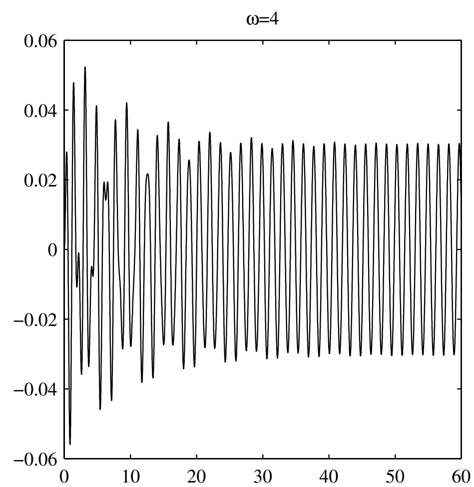




This estimate of the resonant frequency is verified with the calculation

$$\omega_{\text{res}} = \sqrt{\omega_0^2 - 2c^2} = \sqrt{7^2 - 2(0.1)^2} \approx 6.9986.$$

In the figure on the right, careful attention was paid to eliminating transients. In this underdamped case, $T_c = 1/c = 1/0.1 = 10$. Thus, we must go beyond $4T_c = 40$ to eliminate most of the transient behavior. Of course, this extended time interval allows truncation error to propogate, so we set the relative error tolerance of our Matlab solver to 1×10^{-8} to draw the solution in the right figure above, with $\omega = 6.99$ set near the resonant frequency. Note the gain is a small attenuation of the amplitude of the forcing function, $\cos 6.99t$. This makes perfect sense in light of the graph of the gain at the left. Note that the maximum gain is about 0.72, so even near the resonant frequency, we can expect some small degradation of the amplitude of the forcing function. In the figure that follows, $\omega = 4$ was chosen at quite a distance from the resonant frequency.



Note the severe attenuation of the amplitude.

42.

43. (a) Because the current is the derivative of the charge ($I = Q'$), the equation $LI' + RI + (1/C)Q = A\cos\omega t$ becomes $LQ'' + RQ' + (1/C)Q = A\cos\omega t$, or upon dividing both sides by the inductance,

$$Q'' + \frac{R}{L}Q' + \frac{1}{LC}Q = \frac{A}{L}\cos\omega t.$$

The solution we seek is the real part of the complex solution of

$$z'' + \frac{R}{L}z' + \frac{1}{LC}z = \frac{A}{L}e^{i\omega t}.$$

Let $z = ae^{i\omega t}$ represent the steady-state solution. Substituting,

$$\left[(i\omega)^2 + \frac{R}{L}(i\omega) + \frac{1}{LC}\right] ae^{i\omega t} = \frac{A}{L}e^{i\omega t}$$

$$\left[\left(\frac{1}{LC} - \omega^2\right) + \frac{R}{L}i\omega\right] z = \frac{A}{L}e^{i\omega t}.$$

The complex coefficient of z can be written in polar form,

$$\sqrt{\left(\frac{1}{LC} - \omega^2\right)^2 + \frac{R^2}{L^2}\omega^2}\; e^{i\phi(\omega)}\, z = \frac{A}{L}e^{i\omega t},$$

where

$$\cot\phi(\omega) = \frac{\frac{1}{LC} - \omega^2}{\frac{R}{L}\omega}.$$

Thus,

$$z = \frac{A/L}{\sqrt{\left(\frac{1}{LC} - \omega^2\right)^2 + \frac{R^2}{L^2}\omega^2}}\, e^{i(\omega t - \phi(\omega))}.$$

The real part of z,

$$Q = \frac{A/L}{\sqrt{\left(\frac{1}{LC} - \omega^2\right)^2 + \frac{R^2}{L^2}\omega^2}}\; \cos(\omega t - \phi(\omega)),$$

is the solution we seek. The charge will be maximized only if we can maximize

$$G(\omega) = \left[\left(\frac{1}{LC} - \omega^2\right)^2 + \frac{R^2}{L^2}\omega^2\right]^{-1/2}.$$

Taking the derivative,

$$G'(\omega) = -\frac{1}{2}\left[\left(\frac{1}{LC} - \omega^2\right)^2 + \frac{R^2}{L^2}\omega^2\right]^{-3/2}\left[2\left(\frac{1}{LC} - \omega^2\right)(-2\omega) + \frac{2R^2}{L^2}\omega\right]$$

$$= \left[\left(\frac{1}{LC} - \omega^2\right)^2 + \frac{R^2}{L^2}\omega^2\right]^{-3/2}\left[2\omega\left(\frac{1}{LC} - \omega^2\right) - \frac{R^2}{L^2}\omega\right].$$

To find the critical value, set

$$2\omega\left(\frac{1}{LC} - \omega^2\right) - \frac{R^2}{L^2}\omega = 0.$$

Assuming the forcing function has a nonzero frequency,

$$\frac{2}{LC} - 2\omega^2 - \frac{R^2}{L^2} = 0$$

$$\omega^2 = \frac{1}{LC} - \frac{R^2}{2L^2}$$

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{2L^2}}.$$

The careful reader will show that the oscillator is underdamped only if $1/(LC) > R^2/(4L^2)$. The resonant frequency occurs only if $1/(LC) > R^2/(2L^2)$, a bit more than underdamped.

(b) Start by differentiating $LI' + RI + (1/C)Q = A\cos\omega t$, remembering that $Q' = I$.

$$LI'' + RI' + \frac{1}{C}I = -A\omega\sin\omega t,$$

or, upon dividing by the inductance,

$$I'' + \frac{R}{L}I' + \frac{1}{LC}I = -\frac{A\omega}{L}\sin\omega t.$$

Substitute $z = ae^{i\omega t}$ in

$$z'' + \frac{R}{L}z' + \frac{1}{LC}z = -\frac{A\omega}{L}e^{i\omega t}$$

to get

$$\begin{aligned}\left[(i\omega)^2 + \frac{R}{L}(i\omega) + \frac{1}{LC}\right]ae^{i\omega t} &= -\frac{A\omega}{L}e^{i\omega t}\\ \left[\left(\frac{1}{LC} - \omega^2\right) + \frac{R}{L}i\omega\right]z &= -\frac{A\omega}{L}e^{i\omega t}\\ z &= \frac{-A\omega/L}{\left(\frac{1}{LC} - \omega^2\right) + \frac{R}{L}i\omega}\,e^{i\omega t}\\ &= \frac{-A}{\left(\frac{1}{\omega C} - L\omega\right) + Ri}\,e^{i\omega t}.\end{aligned}$$

If we write the denominator of this last expression in polar form

$$\sqrt{\left(\frac{1}{\omega C} - L\omega\right)^2 + R^2}\,e^{i\phi(\omega)},$$

then

$$z = \frac{-A}{\sqrt{\left(\frac{1}{\omega C} - L\omega\right)^2 + R^2}}\,e^{i(\omega t - \phi(\omega))}.$$

The imaginary part of this is the steady-state solution for the current.

$$I = \frac{-A}{\sqrt{\left(\frac{1}{\omega C} - L\omega\right)^2 + R^2}}\,\sin(\omega t - \phi(\omega))$$

Again, the current is maximized by maximizing the amplitude,

$$\frac{-A}{\sqrt{\left(\frac{1}{\omega C} - L\omega\right)^2 + R^2}}.$$

We could proceed as before by taking derivatives, but in this case we can note that the amplitude is maximized when the denominator is minimized.

The denominator's smallest possible value is R, which occurs when

$$\begin{aligned}\frac{1}{\omega C} - L\omega &= 0\\ LC\omega^2 &= 1\\ \omega^2 &= \frac{1}{LC}\\ \omega &= \sqrt{\frac{1}{LC}}.\end{aligned}$$

44.

45. First, $m = 50\,\text{g} = 0.05\,\text{kg}$ and $y = 10\,\text{cm} = 0.1\,\text{m}$. Use Hooke's law to compute the spring constant.

$$k = \frac{F}{y} = \frac{mg}{y} = \frac{(0.05)(9.8)}{0.1} = 4.9\,\text{N/m}.$$

We are given that the damping force is opposite the velocity, with magnitude $0.1v$, and the driving force is $F(t) = 5\cos 4.4t$. Thus, $my'' + \mu y' + ky = F(t)$ becomes

$$0.05y'' + 0.01y' + 4.9y = 5\cos 4.4t,$$

or

$$y'' + 0.2y' + 98y = 100\cos 4.4t.$$

This has characteristic polynomial $P(\lambda) = \lambda^2 + 0.2\lambda + 98$ and zeros $\lambda = -0.1 \pm 9.8990$. Thus, the homogeneous solution is

$$y_h(t) = e^{-0.1t}(C_1 \cos 9.8990t + C_2 \sin 9.8990t).$$

To find the steady-state solution, substitute $z = ae^{i4.4t}$ in $z'' + 0.2z' + 98z = 100e^{i4.4t}$.

$$\begin{aligned}[(4.4i)^2 + 0.2(4.4i) + 98]ae^{i4.4t} &= 100e^{i4.4t}\\ (78.64 + 0.88i)z &= 100e^{i4.4t}.\end{aligned}$$

The coefficient of z has magnitude 78.6449 and phase 0.0112, so

$$\begin{aligned}78.6449e^{0.0112i}z &= 100e^{i4.4t}\\ z &= 1.2716e^{i(4.4t-0.0112)}.\end{aligned}$$

The real part of this, $y_p(t) = 1.2716\cos(4.4t - 0.0112)$, is the steady-state solution. Thus, the solution is

$$\begin{aligned}y(t) &= y_h(t) + y_p(t)\\ &= e^{-0.1t}(C_1\cos 9.8990t + C_2 \sin 9.98990t) + 1.2716\cos(4.4t - 0.0112).\end{aligned}$$

The initial condition $y(0) = 0$ leads to

$$0 = C_1 + 1.2716\cos(-0.0112)$$

and $C_1 = -1.2715$. Differentiating,

$$y'(t) = e^{-0.1t}(-9.8990C_1 \sin 9.8990t + 9.8990C_2 \cos 9.8990t)$$
$$- 0.1e^{-0.1t}(C_1 \cos 9.8990t + C_2 \sin 9.8990t)$$
$$- 5.5950 \sin(4.4t - 0.0112).$$

The initial condition $y'(0) = 0$ leads to

$$0 = 9.8990C_2 - 0.1C_1 - 5.5950 \sin(-0.0112)$$

and $C_2 = -0.0192$. Thus, the solution is

$$y(t) = e^{-0.1t}(-1.2715 \cos 9.8990t - 0.0192 \sin 9.8990t)$$
$$+ 1.2716 \cos(4.4t - 0.0112).$$

Section 4.8