Math 211

Homework #3

February 2, 2001

2.4.3. $y' + (2/x)y = (\cos x)/x^2$

Answer: Compare $y' + (2/x)\,y = (\cos x)/x^2$ with $y' = a(x)x + f(x)$ and note that $a(x) = -2/x$. Consequently, an integrating factor is found with

$$u(x) = e^{\int -a(x)\,dx} = e^{\int 2/x\,dx} = e^{2\ln|x|} = |x|^2 = x^2.$$

Multiply both sides of our equation by the integrating factor and note that the left-hand side of the resulting equation is the derivative of a product.

$$\begin{aligned} x^2\left(y' + \frac{2}{x}\,y\right) &= \cos x \\ x^2y' + 2xy &= \cos x \\ (x^2y)' &= \cos x \end{aligned}$$

Integrate and solve for x.

$$\begin{aligned} x^2y &= \sin x + C \\ y &= \frac{\sin x + C}{x^2} \end{aligned}$$

2.4.7. $(1 + x)y' + y = \cos x$

Answer: Divide both sides by $1 + x$ and solve for y'.

$$y' = -\frac{1}{1+x}y + \frac{\cos x}{1+x}$$

Compare this result with $y' = a(x)y + f(x)$ and note that $a(x) = -1/(1 + x)$. Consequently, an integrating factor is found with

$$u(x) = e^{\int -a(x)\,dx} = e^{\int 1/(1+x)\,dx} = e^{\ln|1+x|} = |1 + x|.$$

If $1 + x > 0$, then $|1 + x| = 1 + x$. If $1 + x < 0$, then $|1 + x| = -(1 + x)$. In either case, if we multiply both sides of our equation by either integrating factor, we arrive at

$$(1 + x)y' + y = \cos x.$$

Check that the left-hand side of this result is the derivative of a product, integrate, and solve for y.

$$\begin{aligned} ((1 + x)y)' &= \cos x \\ (1 + x)y &= \sin x + C \\ y &= \frac{\sin x + C}{1 + x} \end{aligned}$$

2.4.14. $y' = y + 2xe^{2x}, \quad y(0) = 3$

Answer: Compare $y' = y + 2xe^{2x}$ with $y' = a(x)y + f(x)$ and note that $a(x) = 1$. Consequently, an integrating factor is found with

$$u(x) = e^{\int -a(x)\,dx} = e^{\int -1\,dx} = e^{-x}.$$

Multiply both sides of our equation by the integrating factor and note that the left-hand side of the resulting equation is the derivative of a product.

$$\begin{aligned} e^{-x}y' - e^{-x}y &= 2xe^{x} \\ \left(e^{-x}y\right)' &= 2xe^{x} \end{aligned}$$

Integration by parts yields

$$\int 2xe^{x}\,dx = 2xe^{x} - \int 2e^{x} = 2xe^{x} - 2e^{x} + C.$$

Consequently,

$$\begin{aligned} e^{-x}y &= 2xe^{x} - 2e^{x} + C \\ y &= 2xe^{2x} - 2e^{2x} + ce^{x} \end{aligned}$$

The initial condition provides

$$3 = y(0) = 2(0)e^{2(0)} - 2e^{2(0)} + Ce^{0} = -2 + C.$$

Consequently, $C = 5$ and $y = 2xe^{2x} - 2e^{2x} + 5e^{x}$.

2.4.16. $(1 + t^2)y' + 4ty = (1 + t^2)^{-2}, \quad y(1) = 0$

Answer: Solve for y'.

$$y' = -\frac{4t}{1 + t^2}\,y + \frac{1}{1 + t^2}$$

Compare this with $y' = a(t)y + f(t)$ and note that $a(t) = -4t/(1 + t^2)$. Consequently, an integrating factor is found with

$$u(t) = e^{\int -a(t)\,dx} = e^{\int 4t/(1+t^2)\,dt} = e^{2\ln|1+t^2|} = (1 + t^2)^2.$$

Multiply both sides of our equation by the integrating factor and note that the left-hand side of the resulting equation is the derivative of a product.

$$\begin{aligned} (1 + t^2)^2y' + 4t(1 + t^2) &= \frac{1}{1 + t^2} \\ \left((1 + t^2)^2y\right)' &= \frac{1}{1 + t^2} \\ (1 + t^2)^2y &= \tan^{-1} t + C \end{aligned}$$

The initial condition $y(1) = 0$ gives

$$(1 + 1^2)^2(0) = \tan^{-1} 1 + C.$$

Consequently, $C = -\pi/4$ and

$$y = \frac{\tan^{-1} t - \frac{\pi}{4}}{(1 + t^2)^2}.$$

2.4.18. $xy' + 2y = \sin x, \quad y(\pi/2) = 0$

Answer:

Solve $xy' + 2y = \sin x$ for y'.

$$y' = -\frac{2}{x}\,y + \frac{\sin x}{x}$$

Compare this with $y' = a(x)y + f(x)$ and note that $a(x) = -2/x$ and $f(x) = (\sin x)/x$. It is important to note that neither a nor f is continuous at $x = 0$, a fact that will heavily influence our interval of existence.

An integrating factor is found with

$$u(x) = e^{\int -a(x)\,dx} = e^{\int 2/x\,dx} = e^{2\ln|x|} = |x|^2 = x^2.$$

Multiply both sides of our equation by the integrating factor and note that the left-hand side of the resulting equation is the derivative of a product.

$$\begin{aligned} x^2y' + 2xy &= x\sin x \\ \left(x^2y\right)' &= x\sin x \end{aligned}$$

Integration by parts yields

$$\int x\sin x\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C.$$

Consequently,

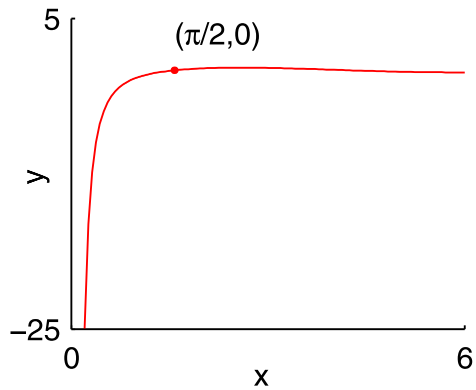
$$\begin{aligned} x^2y &= -x\cos x + \sin x + C, \\ y &= -\frac{1}{x}\cos x + \frac{1}{x^2}\sin x + \frac{C}{x^2}. \end{aligned}$$

The initial condition provides

$$0 = y(\pi/2) = \frac{4}{\pi^2} + \frac{4C}{\pi^2}.$$

Consequently, $C = -1$ and $y = -(1/x)\cos x + (1/x^2)\sin x - 1/x^2$.

We cannot extend any interval to include $x = 0$, as our solution is undefined there. The initial condition $y(\pi/2) = 0$ forces the solution through a point with $x = \pi/2$, a fact which causes us to select $(0, +\infty)$ as the interval of existence. The solution curve is shown in Figure . Note how it drops to negative infinity as x approaches zero from the right.



Solution of $xy' + 2y = \sin x$, $y(\pi/2) = 0$.

2.5.1. A tank contains 100 gal of pure water. At time zero, sugar-water solution containing 0.2 lb of sugar per gal enters the tank at a rate of 3 gal per minute. Simultaneously, a drain is opened at the bottom of the tank allowing sugar-solution to leave the tank at 3 gal per minute. Assume that the solution in the tank is kept perfectly mixed at all times.

Answer: Let $S(t)$ denote the amount of sugar in the tank, measured in pounds. The rate in is $3\text{g/m} \times 0.2\text{lb/g} = 0.6\text{lb/m}$. The rate out is $3\text{g/m} \times S/100\text{lb/g} = 3S/100\text{lb/m}$. Hence

$$\begin{aligned}\frac{dS}{dt} &= \text{rate in} - \text{rate out}\\ &= 0.6 - 2S/100\end{aligned}$$

This linear equation can be solved using the integrating factor $u(t) = e^{3t/100}$ to get the general solution $S(t) = 20 + Ce^{-3t/100}$. Since $S(0) = 0$, the constant $C = -20$ and the solution is $S(t) = 20(1 - e^{-3t/100}$.

(a) $S(20) = 10(1 - e^{-0.6}) \approx 9.038$

(b) $S(t) = 15$ when $e^{-3t/100} = 1 - 15/20 = 1/4$. Taking logarithms this translates to $t = (100 \ln 4)/3 \approx 46.2098$.

(c) As $t \to \infty$ $S(t) \to 20$.

2.5.3. A tank initially contains 100 gal of water in which is dissolved 2 lb of salt. Salt-water solution containing 1 lb of salt for every 4 gal of solution enters the tank at a rate of 5 gal per minute. Solution leaves the tank at the same rate, allowing for a constant solution volume in the tank.

(a) Use an analytic method to determine the eventual salt content in the tank.

Answer: Let $x(t)$ represent the number of pounds of salt in the tank at time t. The rate at which the salt in the tank is changing with respect to time is equal to the rate at which salt enters the tank minus the rate at which salt leaves the tank, i.e.,

$$\frac{dx}{dt} = \text{rate in} - \text{rate out}.$$

In order that the units match in this equation, dx/dt, the rate In, and the rate Out must each be measured in pounds per minute (lb/min).

Solution enters the tank at 5 gal/min, but the concentration of this solution is 1/4 lb/gal. Consequently,

$$\text{rate in} = 5\,\text{gal/min} \times \frac{1}{4}\,\text{lb/gal} = \frac{5}{4}\,\text{lb/min}.$$

Solution leaves the tank at 5 gal/min, but at what concentration? Assuming perfect mixing, the concentration of salt in the solution is found by dividing the amount of salt by the volume of solution, $c(t) = x(t)/100$. Consequently,

$$\text{rate out} = 5\,\text{gal/min} \times \frac{x(t)}{100}\,\text{lb/gal} = \frac{1}{20}x(t)\,\text{lb/min}.$$

As there are 2 lb of salt present in the solution initially, $x(0) = 2$ and

$$\frac{dx}{dt} = \frac{5}{4} - \frac{1}{20}x, \quad x(0) = 2.$$

Multiply by the integrating factor, $e^{(1/20)t}$, and integrate.

$$\begin{aligned} \left(e^{(1/20)t}x\right)' &= \frac{5}{4}e^{(1/20)t} \\ e^{(1/20)t}x &= 25e^{(1/20)t} + C \\ x &= 25 + Ce^{-(1/20)t} \end{aligned}$$

The initial condition $x(0) = 2$ gives $C = -23$ and

$$x(t) = 25 - 23e^{-(1/20)t}.$$

Thus, the concentration at time t is given by

$$c(t) = \frac{x(t)}{100} = \frac{25 - 23e^{-(1/20)t}}{100},$$

and the eventual concentration can be found by taking the limit as $t \to +\infty$.

$$\lim_{t \to +\infty} \frac{25 - 23e^{-(1/20)t}}{100} = \frac{1}{4}\,\text{lb/gal}$$

Note that this answer is quite reasonable as the concentration of solution entering the tank is also 1/4 lb/gal.

(b) Use a numerical solver to determine the eventual salt content in the tank and compare your approximation with the analytical solution found in part (a).

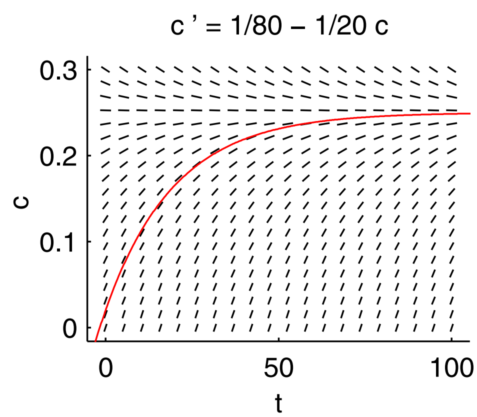
Answer: We found it convenient to manipulate our original differential equation before using our solver. The key idea is simple: we want to sketch the concentration $c(t)$, not the salt content $x(t)$. However,

$$c(t) = \frac{x(t)}{100} \qquad \text{or} \qquad x(t) = 100c(t).$$

Consequently, $x'(t) = 100c'(t)$. Substituting these into our balance equation gives

$$\begin{aligned} x' &= \frac{5}{4} - \frac{1}{20}x, \\ 100c' &= \frac{5}{4} - \frac{1}{20}(100c), \\ c' &= \frac{1}{80} - \frac{1}{20}c, \end{aligned}$$

with $c(0) = x(0)/100 = 2/100 = 0.02$. The numerical solution of this ODE is presented in Figure . Note how the concentration approaches 0.25 lb/gal.



Solution of $c' = 1/80 - (1/20)c$, $c(0) = 0.02$.

2.5.6. A tank initially contains 100 gal of a salt-water solution containing 0.05 lb of salt for each gallon of water. At time zero, pure water is poured into the tank at a rate of 2 gal per minute. Simultaneously, a drain is opened at the bottom of the tank that allows salt-water solution to leave the tank at a rate of 3 gal per minute. What will be the salt content in the tank when precisely 50 gal of salt solution remain?

Answer: The volume in the tank is decreasing at 1 g/m, so the volume is $V(t) = 100 - t$. There is no sugar coming in, and the rate out is 3g/m $\times\ S(t)/V(t)$. Hence the differential equation is

$$\frac{dS}{dt} = \frac{-3S}{100 - t}.$$

This equation is linear and homogenous. It can be solved by separating variables. The general solution is $S(t) + A(100 - t)^3$. Since $S(0) = 100 \times 0.05 = 5$, we see that $A = 5 \times 10^{-6}$, and the solution is $S(t) = 5 \times 10^{-6} \times (100 - t)^3$.
When $V(t) = 100 - t = 50$g,

$$S(t) = 5 \times 10^{-6} \times 50^3 = 0.625\text{lb}.$$

2.5.8. Suppose that a solution containing a drug enters an organ at the rate $a\,\text{cm}^3/\text{s}$, with drug concentration $\kappa\ \text{g/cm}^3$. Solution leaves the organ at a slower rate of $b\,\text{cm}^3/\text{s}$. Further, the faster rate of infusion causes the organ's volume to increase with time according to $V(t) = V_0 + rt$, with V_0 its initial volume. If there is no initial quantity of the drug in the organ, show that the concentration of the drug in the organ is given by

$$c(t) = \frac{a\kappa}{b+r}\left[1 - \left(\frac{V_0}{V_0+rt}\right)^{(b+r)/r}\right].$$

Answer: Let $x(t)$ represent the amount of drug in the organ at time t. The rate at which the drug enters the organ is

$$\text{rate In} = a\,\text{cm}^3/\text{s} \times \kappa\ \text{g/cm}^3 = a\kappa\ \text{g/s}.$$

The rate at which the drug leaves the organ equals the rate at which fluid leaves the organ, multiplied by the concentration of the drug in the fluid at that time. Hence,

$$\text{rate out} = b\,\text{cm}^3/\text{s} \times \frac{x(t)}{V_0+rt}\ \text{g/cm}^3 = \frac{b}{V_0+rt}x(t)\ \text{g/s}.$$

Consequently,

$$\frac{dx}{dt} = a\kappa - \frac{b}{V_0+rt}x.$$

The integrating factor is

$$u(t) = e^{\int b/(V_0+rt)\,dt} = e^{(b/r)\ln(V_0+rt)} = (V_0+rt)^{b/r}.$$

Multiply by the integrating factor and integrate.

$$\begin{aligned}
\left((V_0+rt)^{b/r}x\right)' &= a\kappa(V_0+rt)^{b/r}\\
(V_0+rt)^{b/r}x &= \frac{a\kappa}{r(b/r+1)}(V_0+rt)^{b/r+1} + L\\
x &= \frac{a\kappa}{b+r}(V_0+rt) + L(V_0+rt)^{-b/r}
\end{aligned}$$

No drug in the system initially gives $x(0) = 0$ and $L = -a\kappa V_0^{b/r+1}/(b+r)$. Consequently,

$$\begin{aligned}
x &= \frac{a\kappa}{b+r}(V_0+rt) - \frac{a\kappa V_0^{b/r+1}}{b+r}(V_0+rt)^{-b/r},\\
x &= \frac{a\kappa}{b+r}(V_0+rt)\left[1 - V_0^{b/r+1}(V_0+rt)^{-b/r-1}\right],\\
x &= \frac{a\kappa}{b+r}(V_0+rt)\left[1 - \left(\frac{V_0}{V_0+rt}\right)^{b/r+1}\right].
\end{aligned}$$

The concentration is found by dividing $x(t)$ by $V(t) = V_{+}rt$. Consequently,

$$c(t) = \frac{a\kappa}{b+r}\left[1 - \left(\frac{V_0}{V_0+rt}\right)^{(b+r)/r}\right].$$

2.7.1. Is the initial value problem $y' = 4 + y^2, \quad y(0) = 1$ guaranteed a unique solution by the hypotheses of Theorem 7.12? Justify your answer.

Answer: The right hand side of the equation is $f(t, y) = 4+y^2$. f is continuous in the whole plane. Its partial derivative $\partial f/\partial y = 2y$ is also conitnous on the whole plane. Hence the hypotheses are satisfied and the theorem guarantees a unique solution.

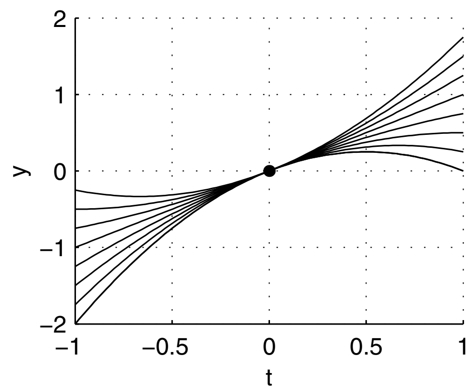
2.7.2. Is the initial value problem $y' = \sqrt{y}, \quad y(4) = 0$ guaranteed a unique solution by the hypotheses of Theorem 7.12? Justify your answer.

Answer: The right hand side of the equation is $f(t, y) = \sqrt{y}$. f is defined only where $y \geq 0$, and it is continuous there. Our initial condition is at $y_0 = 0$, and $t_0 = 4$. There is no rectangle containing (t_0, y_0) where $f(t, y)$ is defined and continuous. Consequently the hypotheses of the theorem are not satisfied.

2.7.8. $ty' = 2y - t, \quad y(0) = 2$

(i) Find the general solution of the differential equation. Sketch several members of the family of solutions portrayed by the general solution.

Answer: The equation is linear. The general solution is $y(t) = t + 2Ct^2$.



(ii) Show that there is no solution satisfying the given initial condition. Explain why this lack of solution does not contradict the existence theorem.

Answer: Since the general solution is $y(t) = t + 2Ct^2$, every solution satisfies $y(0) = 0$. There is no solution with $y(0) = 2$. If we put the equation into normal form

$$\frac{dy}{dt} = \frac{2y - t}{t},$$

we see that the right hand side $f(t, y) = \dfrac{2y - t}{t}$ fails to be continuous at $t = 0$. Consequently the hypotheses of the existence theorem are not satisfied.

2.7.29. Suppose that y is a solution to the initial value problem

$$y' = (y^2 - 1)e^{ty} \quad \text{and} \quad y(1) = 0.$$

Show that $-1 < y(t) < 1$ for all t for which y is defined.

Answer: Notice that the right hand side of the equation is $f(t, y) = (y^2 - 1)e^{ty}$. f is continuous on the whole plane. Its partial derivative $\partial f/\partial y = 2ye^{ty} + t(y^2 - 1)e^{ty}$ is also continuous on the whole plane. Thus the hypotheses of the uniqueness theorem are satisfied. By direct substitution we discover that $y_1(t) = -1$ and $y_2(t) = 1$ are both solutions to the differential equation. If y is a solution and satisfies $y(1) = 0$, then $y_1(1) < y(1) < y_2(1)$. By the uniqueness theorem we must have $y_1(t) < y(t) < y_2(t)$ for all t for which y is defined. Hence $-1 < y(t) < 1$ for all t for which y is defined.

M2.8.

Answer: We derive the model in terms of $P(t)$, the amount of pollutant in the lake, measured in km^3. The rate in is p km^3/h. the rate out is $(r_i + p) \times P/V$. Hence the model is

$$P' = p - \frac{(r_i + p)P}{V}.$$

To put this in terms of the concentration we notice that $c = P/V$. Since the volume is constant the derivative is $c' = P'/V$. Hence

$$\begin{aligned} c' &= \frac{1}{V} P' \\ &= \frac{p}{V} - \frac{(r_i + p)P}{V^2} \\ &= \frac{p}{V} - \frac{(r_i + p)c}{V}. \end{aligned}$$

Hence we have

$$c' + \frac{(r_i + p)c}{V} = \frac{p}{V}.$$

M2.9.

Answer:

(a) From problem 8, one obtains the differential equation $c' = p/V - ((p + r_i)/V)c$. Substituting $p = 2$, $r = 50$, and $V = 100$ and simplifying gives $c' = 1/50 - (26/50)c$. So, write `c' = 1/50 - (26/50)*c` in the dialog for `dfield5`, let t run from 0 to 5 and c from -0.02 to 0.05, and obtain Figure 1.

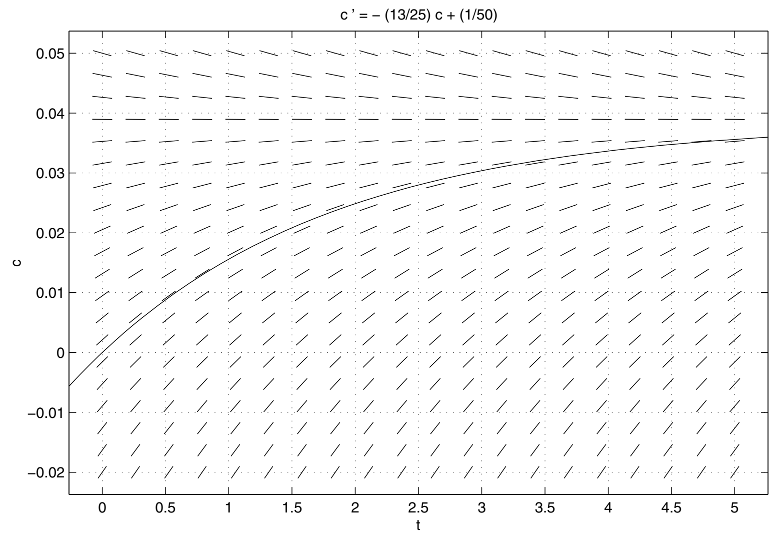


Figure 1.

(b) Using the zoom box technique, one can obtain a picture much like that given in Figure 2, showing that when $t = 1.4101$, then $c = 0.02$, which is 2%.

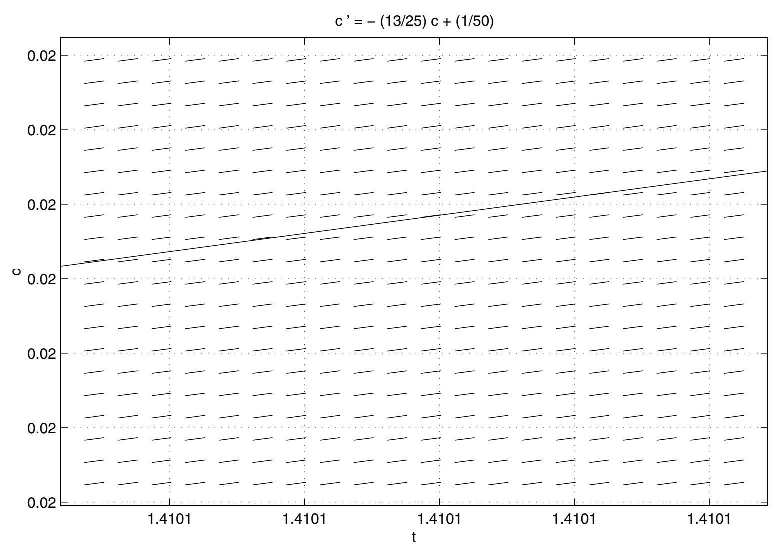


Figure 2.

(c) The limiting value of c can be found either by explicitly solving the differential equation or by looking at long-term behaviour of the differential equation in `dfield5`. If one solves the differential equation, one should obtain

$$c(t) = 1/26 - 1/26e^{-13t/25}.$$

Taking the limit $\lim_{t\to\infty} c(t)$, the exponential term goes to zero, leaving $1/26$. Hence the limiting value of $c(t)$ is $1/26$, or about 3.85%. Using `dfield5` and setting the parameter t to go from 0 to 50, for instance, one can see that the value of c approaches 0.0385, as expected. To find the approximate value of t when $c = 0.035$ — that is, 3.5% — use the zoom box technique to obtain a picture like Figure 3. From this, one can see that $t = 4.63$.

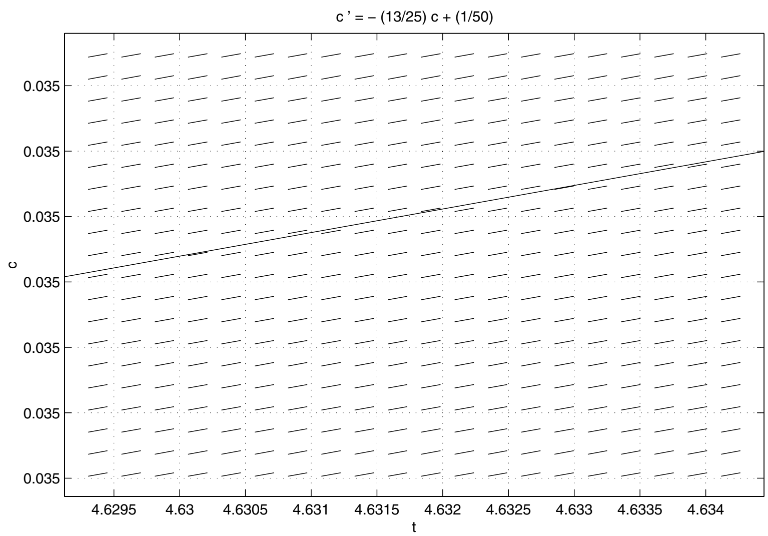


Figure 3.

M2.10.

Answer: This problem can be answered analytically. Since there is no input of poluttant, we now have $p = 0$ and the model equation changes to

$$c' = -\frac{50}{100}c = -\frac{c}{2}.$$

The solution to this linear, homogenous equation is $c(t) = c_0 e^{-t/2}$. For $c_0 = 0.035$, we get $c(t) = 0.02$ when $t = 2\ln(7/4) \approx 1.1192$.

We get the same answer (approximately) using `dfield5`. See Figure 4.

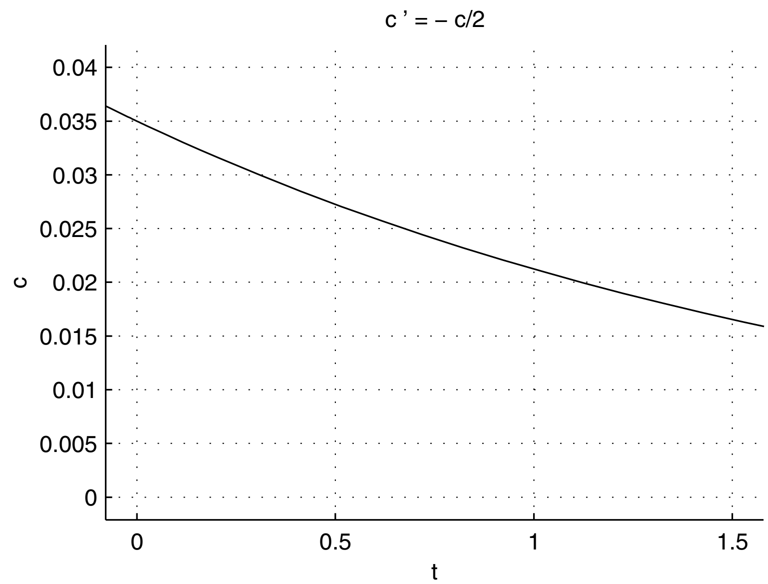


Figure 4.

M2.11.

Answer:

(a) In the `dfield5` setup dialog, enter `c' = 1/50 - ((2 + r)/100)*c` and in one parameter box, enter `r = 50 + 20*cos(2*pi*(t - 1/3))`. In Figure 5 presented below, the initial conditions given were $t = 0$, $c = 0$; $t = 0$, $c = 0.01$; $t = 0$, $c = 0.02$; $t = 0$, $c = 0.03$; $t = 0$, $c = 0.04$. As one can see, each solution tends towards a single oscillatory pattern.

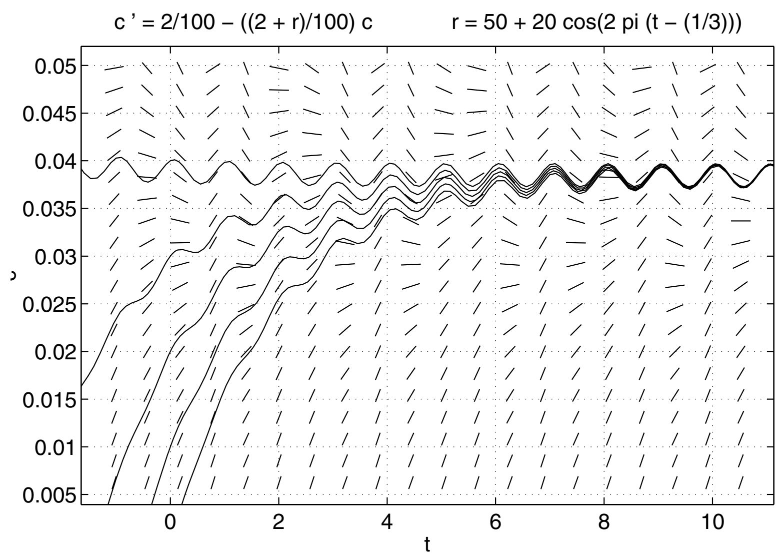


Figure 5.

(b) In Figure 6 below, the t varies from 10 to 11, representing the passage of time over a single year. As one can see, the concentration reaches a maximum at $t = 10.07$, which is shortly before the beginning of February.

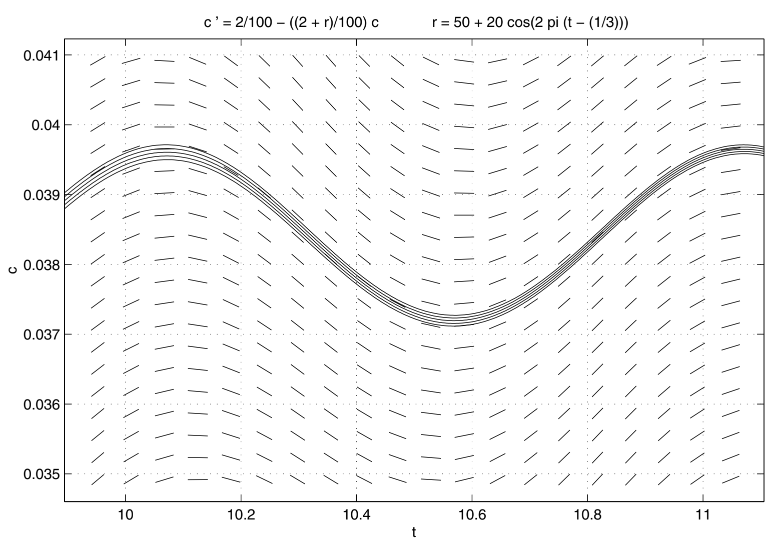


Figure 6.

M2.25.

Answer: In Figure 7 below, the circle indicates the point $t = 4, x = 0$, and the four curves are given by initial conditions $t = 0, x = 0.728$; $t = 0, x = 0.7285$; $t = 0, x = 0.728989$ (the one which hits the target); and $t = 0, x = 0.7295$.

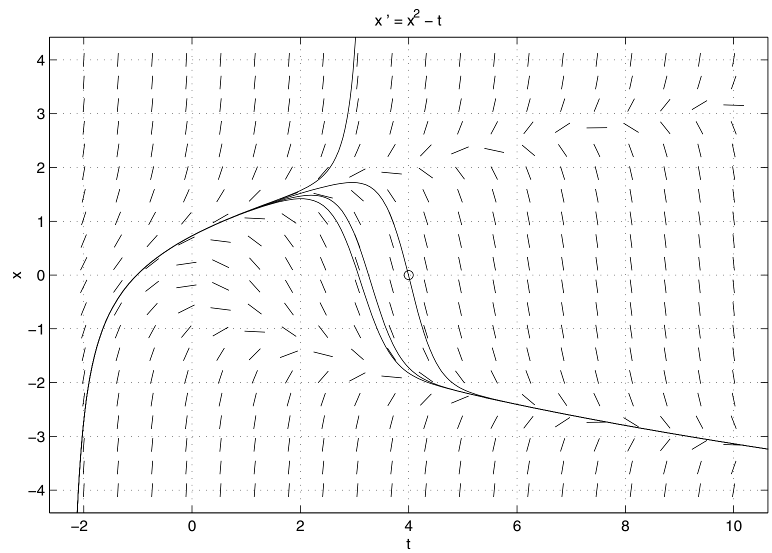


Figure 7.

M2.27.

Answer: In Figure 8 below, the circle indicates the point $t = 5, x = 0$, and the four curves are given by the initial conditions $t = 0, x = -3.2$; $t = 0, x = -3.232$; $t = 0, x = -3.2320905$ (the one which hit the target); and $t = 0, x = -3.3$.

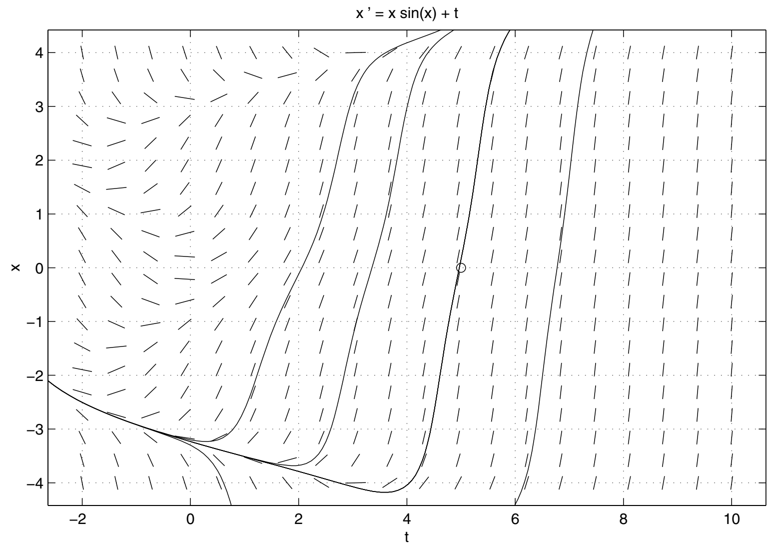


Figure 8.