Project #1
The Koebe-Bieberbach Theorem

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The following is known as the Koebe-Bieberbach theorem.

Theorem: Suppose that f is a holomorphic function on the unit disk **D**, which is injective and satisfies $f(0) = 0$ and $f'(0) = 1$. Then its image $f(\mathbf{D})$ contains the disk $D_{1/4}(0)$.

The project is to prove this theorem and to present the proof and related comments in a well written paper.

The proof

You will find an outline of a proof, together with several hints, in Problem #1 on page 108 of our textbook. Even with the hints, this is a bit hard to follow, so we will provide some additional help keyed to the parts of Problem #1.

(a) and (b) These are examples to show what happens when the hypotheses of the theorem are not satisfied. For part (b), think about the solutions to the equation $e^w = 0$.

(c) The hint is a little terse. Let γ_ρ be the curve parameterized by $\theta \to h(\rho e^{i\theta})$ for $0 \le \theta \le 2\pi$. Since h is injective, this is a simple closed curve. According to the Jordan Curve theorem, γ_ρ separates the plane into two components, one bounded and one unbounded. Since h has a pole at 0, the image $h(D_\rho(0) - 0)$ must be the unbounded component. Thus we want to compute the area of the bounded component, which we will call G_ρ. According to the Gauss-Green theorem, this is given by

$$\text{Area}(G_\rho) = \int_{G_\rho} du \wedge dv = \int_{\partial G_\rho} u \, dv = \int_{\gamma_\rho} u \, dv,$$

since $\gamma_\rho = \partial G_\rho$. The computation can be made somewhat easier if we use complex notation, letting $w = u + iv$. Then $d\overline{w} \wedge dw = 2i \, du \wedge dv$, so

$$\text{Area}(G_\rho) = \frac{1}{2i} \int_{G_\rho} d\overline{w} \wedge dw = \frac{1}{2i} \int_{\gamma_\rho} \overline{w} \, dw.$$

This last integral can be evaluated using the parameterization of γ_ρ and the series expansion for h. The double sum coming from the product may be daunting, but a large proportion of the terms turn out to be equal to 0.

(d) Noticing that g is an odd function will help in what follows.

(e) Notice that $1/g$ has a simple pole at the origin. Thus the authors do not really mean for you to find a power series for $1/g$. First find the (first few terms in) the power series for g in terms of those of f. Then write

$$\frac{1}{g(z)} = \frac{1}{z} + G(z),$$

and find the first few coefficients of the power series of G.

(f) and (g) No additional hints are needed.

The project report

You are to write up the finished project as though it were a section of a book aimed at yourself and your fellow students. Our textbook is a good model. This means that you are to provide discussion of the result, including motivation and examples that illustrate the result and the importance of the hypotheses. Your proof should be presented in a logical chain, and may proceed through lemmas and propositions that lead to the final result. It is not necessary that you follow the order of the parts in Problem #1.

You should be aware that 20% of your grade will be based on the quality of your presentation.

You can find a set of guidelines for writing reports in the document *Math 211 Project Reports*, which is available at

http://www.owlnet.rice.edu/~math211/Math211ProjGuides.pdf.

Just ignore those aspects that apply specifically to Math 211.