Project #2
The Poincaré Disk

March 24, 2004

The Poincaré disk is the unit disk $\mathbb{D}$ supplied with the Poincaré metric. Your task is to discuss the properties of the Poincaré disk, and in particular those of the Poincaré metric.

The main reference

The Poincaré metric is discussed in Problem # 3 on page 256 of our textbook. You will also need the material on the Schwarz-Pick lemma from Exercise # 13 on page 251. In most cases, the hints contained in the problem and the exercise should be enough for you to complete each step. The hardest step is probably the converse in part (b) of Problem # 3. If you have difficulty with this look at Problem #1.

Other information and tasks

As pointed out in the book, the Poincaré metric is an example of a Riemannian metric. Riemannian metrics provide a way of assigning a length to a curve. In this case, if $\gamma : [a, b] \to \mathbb{D}$ is a curve parameterization, then the (Poincaré) length of γ is

$$L(\gamma) = \int_\gamma \frac{|dz|}{1 - |z|^2} = \int_a^b \frac{|\gamma'(t)|\, dt}{1 - |\gamma(t)|^2}.$$

Of course, the Poincaré length is not the same as the Euclidean length we have been using up to now.

A *geodesic* is a curve γ which minimizes the distance between any two points on γ. The unit disk $\mathbb{D}$ equiped with the Poincaré metric is called the *Poincaré disk.* We will call a geodesic in the Poincaré disk a *hyperbolic line.* With this definition of line, the Poincaré disk provides an example of a non-Euclidean geometry. Verify this by completing the following tasks:

1. Show that an automorphism of the disk maps a hyperbolic line into another hyperbolic line.
2. Identify the hyperbolic lines.
3. Verify the following incidence relations:
   a) Through any two points there is one and only one hyperbolic line.
   b) Given a hyperbolic line and a point not on that line there are infinitely many lines through the point that are parallel to the line. Remember that parallel lines are defined to be lines that do not intersect.

In this regard you may be interested in playing with the applet Noneuclid. You can find it on the web at http://cs.unm.edu/ joel/NonEuclid/.

The project report

You are to write up the finished project as though it were a section of a book aimed at yourself and your fellow students. Our textbook is a good model. This means that you are to provide discussion of the result, including motivation and examples that illustrate the result and the importance of the hypotheses. Your proofs should be presented in a logical chain, and may proceed through lemmas and propositions that lead to the final result.

You can find a set of guidelines for writing reports in the document *Math 211 Project Reports*, which is available at

http://www.owlnet.rice.edu/~math211/Math211ProjGuides.pdf.

Just ignore those aspects that apply specifically to Math 211.