

Math 211

Lecture #15

October 19, 2000

Definition of Determinant

The general $n \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$\det(A) = \sum_{\sigma} (-1)^{\sigma} a_{1\sigma_1} \cdot a_{2\sigma_2} \cdot \cdots \cdot a_{n\sigma_n}$$

Definition of Determinant

- Each summand of the definition is the product of n entries, one from each row, and one from each column.
- The definition is the sum of $n!$ terms. It does not provide an effective way to compute the determinant.
- One exception: The determinant of an upper triangular matrix is the product of the diagonal terms.

Row Operations and Determinants

If B is obtained from A by

- adding a multiple of one row to another,
$$\det(B) = \det(A).$$
- interchanging two rows,
$$\det(B) = -\det(A).$$
- multiplying a row by $c \neq 0$,
$$\det(B) = c \det(A).$$

Example

$$A = \begin{pmatrix} -5 & 0 & 0 \\ -25 & -4 & -14 \\ 65 & 7 & 17 \end{pmatrix}$$

$$\det(A) = -150$$

Properties

- If A has two equal rows, then $\det(A) = 0$.
- $\det(A^T) = \det(A)$.
- If A has two equal columns, then $\det(A) = 0$.

Column Operations and Determinants

If B is obtained from A by

- adding a multiple of one column to another,
$$\det(B) = \det(A).$$
- interchanging two columns,
$$\det(B) = -\det(A).$$
- multiplying a column by $c \neq 0$,
$$\det(B) = c \det(A).$$

Expansion by a Row

Definition: The ij -minor of an $n \times n$ matrix A is the $(n - 1) \times (n - 1)$ matrix A_{ij} obtained from A by deleting the i^{th} row and the j^{th} column.

With this definition we can prove that

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(A_{ij}).$$

- This is called **expansion by the i^{th} row.**

Expansion by a Column

We can also expand by a column.

$$\det(A) = \sum_{i=1}^n (-1)^{i+j} a_{ij} \det(A_{ij}).$$

- This is called **expansion by the j^{th} column**.

Example

$$A = \begin{pmatrix} -5 & -6 & 0 \\ 3 & 4 & 0 \\ -8 & -16 & 9 \end{pmatrix}$$

$$\begin{aligned} \det(A) &= 9 \cdot \det \begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix} \\ &= 9 \cdot (-2) \\ &= -18 \end{aligned}$$

Expansion by row

Expansion by column

Example

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & -2 \\ -2 & -1 & 1 & 1 \\ 2 & 2 & 1 & 1 \end{pmatrix}$$

$$\det(A) = 1.$$

Expansion by row

Expansion by column

Example

$$A = \begin{pmatrix} 3 & -1 & 0 & 1 \\ 12 & -6 & 0 & 5 \\ 32 & -15 & -3 & 13 \\ 18 & -10 & -1 & 8 \end{pmatrix}$$

$$\det(A) = -1.$$

Determinants and Bases

Proposition: A collection of n vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ in \mathbf{R}^n is a basis for \mathbf{R}^n if and only if

$$\det([\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_n]) \neq 0.$$

Systems of Differential Equations

Example: SIR model of the spread of infectious disease. Assume:

- The disease is of short duration and rarely fatal.
- The disease spreads through human contact.
- Recovered individuals are immune.

SIR Model

- Three subpopulations; susceptible, $S(t)$, infecteds, $I(t)$, and recovered, $R(t)$

$$S' = -aSI$$

$$I' = aSI - bI$$

$$R' = bI.$$

- $N = S + I + R$ is constant.

SIR Model

- New variables $s = S/N$, $i = I/N$, and $r = R/N$.

$$s' = -A si$$

$$i' = A si - bi$$

- $A = Na$.
- $s + i + r = 1$, so $r = 1 - s - i$.
- MATLAB& pp1ane5

General System in 2D

$$x' = f(t, x, y)$$

$$y' = g(t, x, y)$$

- Example:

$$x' = y$$

$$y' = -x$$

- Solution: $x(t) = \sin t$ & $y(t) = \cos t$
 - ◇ Verify by direct substitution.

General System in Higher D

$$x'_1 = f_1(t, x_1, x_2, \dots, x_n)$$

$$x'_2 = f_2(t, x_1, x_2, \dots, x_n)$$

$$\vdots = \quad \quad \quad \vdots$$

$$x'_n = f_n(t, x_1, x_2, \dots, x_n)$$

Vector Notation

Set $u_1(t) = x(t)$ & $u_2(t) = y(t)$.

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned} \Leftrightarrow \mathbf{u}' = \begin{pmatrix} u_2 \\ -u_1 \end{pmatrix}$$

Set $\mathbf{F}(t, \mathbf{u}) = (f(t, u_1, u_2), g(t, u_1, u_2))^T$

$$\begin{aligned} x' &= f(t, x, y) \\ y' &= g(t, x, y) \end{aligned} \Leftrightarrow \mathbf{u}' = \mathbf{F}(t, \mathbf{u})$$

Vector Notation

Set

$$\mathbf{x}(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$$

$$\mathbf{f}(t, \mathbf{x}) = (f_1(t, \mathbf{x}), f_2(t, \mathbf{x}), \dots, f_n(t, \mathbf{x}))^T.$$

- The general system can be written

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}).$$

Initial Value Problem

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

- Each component of $\mathbf{x}(t_0)$ must be specified.
- Example

$$\begin{array}{ll} x' = y & x(0) = 2 \\ y' = -x & y(0) = 13 \end{array} \quad \text{with}$$

Reduction of Higher Order Equation to a System

For any higher order equation there is a first order system which is equivalent to it, in the sense that solutions of the system lead easily to solutions of the equation, and vice versa.

Example of Reduction

- Third-order equation: $y''' + 2yy' = 3 \cos t$
- Set $x_1 = y$, $x_2 = y'$, and $x_3 = y''$.
- Then

$$x_1' = x_2$$

$$x_2' = x_3$$

$$x_3' = 3 \cos t - 2x_1x_2$$

Geometric Interpretation of Solutions

- pp1ane5
- Component plot
- Parametric plot
- Phase plane
- 3-D plot
- Composite plot