

Math 211

Review for the
Second Exam

November 12, 2002

Method of Solution for $A\mathbf{x} = \mathbf{b}$

There are four steps:

1. Use the augmented matrix $M = [A, \mathbf{b}]$.
2. Use row operations to reduce the augmented matrix to row echelon form.
3. Write down the simplified system.
4. Backsolve.
 - ♦ Assign arbitrary values to the free variables.
 - ♦ Backsolve for the pivot variables.

[Return](#)

The Solution Set of $A\mathbf{x} = \mathbf{b}$

- The solution set is the set of all vectors that satisfy $A\mathbf{x} = \mathbf{b}$.
 - ♦ A solution set is best described by giving a parametric presentation. This is provided automatically by the method of elimination and backsolving.
- In the special case when $\mathbf{b} = \mathbf{0}$, we have the homogeneous equation $A\mathbf{x} = \mathbf{0}$, and the solution set is called the *nullspace* of A , denoted by $\text{null}(A)$.
 - ♦ A nullspace is best described by giving a basis.

Basis of $\text{null}(A)$

Definition: A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots$, and \mathbf{v}_k form a *basis* of $\text{null}(A)$ if

1. $\text{null}(A) = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$
 2. $\mathbf{v}_1, \mathbf{v}_2, \dots$, and \mathbf{v}_k are linearly independent.
- Bases are not unique, but every basis of $\text{null}(A)$ has the same number of elements.
 - The number of elements in a basis is the *dimension* of $\text{null}(A)$.

Return

How Do We Know if $\mathbf{w} \in \text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$?

1. Form the matrix $M = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k]$ which has the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots$, and \mathbf{v}_k as its columns.
2. Solve the system $M\mathbf{a} = \mathbf{w}$.
 - a. If there are no solutions, \mathbf{w} is *NOT* in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$.
 - b. If there is a solution $\mathbf{a} = (a_1, a_2, \dots, a_k)^T$, then

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_k\mathbf{v}_k$$

is in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$.

Return

Product

When are $\mathbf{v}_1, \mathbf{v}_2, \dots$, and \mathbf{v}_k Linearly Independent?

1. Form the matrix $M = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k]$ which has the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots$, and \mathbf{v}_k as its columns.
2. Find the nullspace, $\text{null}(M)$.
 - a. If $\text{null}(M) = \{\mathbf{0}\}$, the vectors are linearly independent.
 - b. If $\mathbf{a} \in \text{null}(M)$, and $\mathbf{a} = (a_1, a_2, \dots, a_k)^T \neq \mathbf{0}$, then

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_k\mathbf{v}_k = \mathbf{0}$$

and the vectors are linearly dependent.

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Product

$n \times n$ Matrices

- An $n \times n$ matrix A is *invertible* if there is an $n \times n$ matrix B such that $AB = BA = I$. The matrix $A^{-1} = B$ is called the *inverse* of A .
- The $n \times n$ matrix A is *nonsingular* if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for any right hand side \mathbf{b} .

[Return](#)

Determinants

The determinant of an $n \times n$ matrix A can be computed using:

- Expansion by the i^{th} row:

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(A_{ij}).$$

- Expansion by the j^{th} column:

$$\det(A) = \sum_{i=1}^n (-1)^{i+j} a_{ij} \det(A_{ij}).$$

- Row or column operations to simplify the matrix.

More Properties

- If A has two equal rows, then $\det(A) = 0$.
- If A has a row of all zeros, then $\det(A) = 0$.
- If A has two equal columns, then $\det(A) = 0$.
- If A has a column of all zeros, then $\det(A) = 0$.

[Return](#)

Equivalent Properties

If A is an $n \times n$ matrix, the following are equivalent:

- $\det(A) \neq 0$.
- A is non-singular.
- A is invertible.
- The equation $Ax = b$ has a unique solution for any right hand side b .
- $\text{null}(A)$ is trivial, i.e., $\text{null}(A) = \{0\}$.
 - This means that the only solution to the homogeneous equation $Ax = 0$ is 0 , the zero vector.

Systems of Differential Equations

- $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$, where $\mathbf{x} \in \mathbf{R}^n$, $t \in I = (a, b)$.
- Initial value problem: $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$ with $\mathbf{x}(t_0) = \mathbf{x}_0$
- Existence and uniqueness.
- Representation of solutions:
 - Component plots.
 - Curves in phase space. Parametric plots, $t \rightarrow \mathbf{x}(t)$.
- Reduction of higher order systems to first order systems.

Autonomous Systems

- $\mathbf{x}' = \mathbf{f}(\mathbf{x})$.
- Uniqueness in phase space.
- \mathbf{x}_0 is an *equilibrium point* if $\mathbf{f}(\mathbf{x}_0) = 0$.
- $\mathbf{x}(t) = \mathbf{x}_0$ is the corresponding *equilibrium solution*.
- In phase space, an equilibrium solution plots as a point.
- Nullclines.

Homogeneous Linear Systems

- $\mathbf{x}' = A\mathbf{x}$, A is an $n \times n$ matrix.
- Solution strategy: look for a fundamental set of solutions, i.e., n linearly independent solutions
- λ is an *eigenvalue* of A if there is a nonzero vector \mathbf{v} such that $A\mathbf{v} = \lambda\mathbf{v}$. If λ is an eigenvalue of A , then any vector \mathbf{v} such that $A\mathbf{v} = \lambda\mathbf{v}$ is called an *eigenvector associated with λ* .
 - ♦ If λ an eigenvalue of A , and \mathbf{v} is an associated nonzero eigenvector, then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution to $\mathbf{x}' = A\mathbf{x}$.

Finding Eigenvalues and Eigenvectors

- λ is an eigenvalue of $A \Leftrightarrow \det(A - \lambda I) = 0$.
 - ♦ $p(\lambda) = \det(A - \lambda I)$ is called the *characteristic polynomial* of A .
- \mathbf{v} is an eigenvector associated with the eigenvalue $\lambda \Leftrightarrow \mathbf{v} \in \text{null}(A - \lambda I)$.
 - ♦ $\text{null}(A - \lambda I)$ is called the *eigenspace* of λ .

Procedure to Solve $\mathbf{x}' = A\mathbf{x}$

- Find the eigenvalues of A , i.e., the roots of $\det(A - \lambda I) = 0$.
- For each eigenvalue λ find the eigenspace, i.e., $\text{null}(A - \lambda I)$.
- If λ is a real eigenvalue and \mathbf{v} is an associated nonzero eigenvector, $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a real solution.

Return

Procedure (cont.)

- If λ and $\bar{\lambda}$ are a complex conjugate pair of eigenvalues, and \mathbf{w} is a nonzero eigenvector associated with λ , then $\mathbf{z}(t) = e^{\lambda t}\mathbf{w}$ and $\bar{\mathbf{z}}$ is a linearly independent pair of complex valued solutions.
 - ♦ $\mathbf{x}(t) = \text{Re}(\mathbf{z}(t))$ and $\mathbf{y}(t) = \text{Im}(\mathbf{z}(t))$ are a linearly independent pair of real valued solutions.
- Show that n of these are linearly independent, if we can.

The Degenerate Planar Case

- Find the (only) eigenvalue λ_1 .
- Find an eigenvector $\mathbf{v}_1 \neq \mathbf{0}$.
- Find \mathbf{v}_2 with $(A - \lambda_1 I)\mathbf{v}_2 = \mathbf{v}_1$. To do so:
 - ♦ Start with any vector \mathbf{w} not a multiple of \mathbf{v}_1
 - ♦ Then $(A - \lambda_1 I)\mathbf{w} = a\mathbf{v}_1$ with $a \neq 0$.
 - ♦ Set $\mathbf{v}_2 = \frac{1}{a}\mathbf{w}$. \mathbf{v}_2 is not a multiple of \mathbf{v}_1 .
- $\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1$ and $\mathbf{x}_2(t) = e^{\lambda_1 t}[\mathbf{v}_2 + t\mathbf{v}_1]$ form a fundamental set of solutions.

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Planar Systems

- Distinct real eigenvalues $\lambda_1 < \lambda_2$.
 - ♦ $\lambda_1 < 0 < \lambda_2$ — saddle point.
 - ♦ $\lambda_1 < \lambda_2 < 0$ — nodal sink.
 - ♦ $0 < \lambda_1 < \lambda_2$ — nodal source.
- Complex conjugate eigenvalues $\lambda = \alpha + i\beta$ and $\bar{\lambda} = \alpha - i\beta$.
 - ♦ $\alpha = \text{Re}(\lambda) = 0$ — center.
 - ♦ $\alpha = \text{Re}(\lambda) < 0$ — spiral sink.
 - ♦ $\alpha = \text{Re}(\lambda) > 0$ — spiral source.
- The trace-determinant plane.

Product of a Matrix with a Vector

- The *product* of a matrix A and a vector \mathbf{x} is the linear combination of the columns of A with the elements of \mathbf{x} as coefficients.

- Example:

$$\begin{pmatrix} 3 & -4 & 5 \\ -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} 13 \\ -5 \\ 23 \end{pmatrix} \\ = 13 \begin{pmatrix} 3 \\ -1 \end{pmatrix} + (-5) \begin{pmatrix} -4 \\ 2 \end{pmatrix} + 23 \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

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