

# Math 211

Lecture #30

The Exponential of a Matrix

November 6, 2002

## Repeated Eigenvalues – Example 1

$$A = \begin{pmatrix} -5 & -10 & 6 \\ 8 & 19 & -12 \\ 12 & 30 & -19 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 3)(\lambda + 1)^2$
- $\lambda_1 = -3$  : Eigenspace has dimension 1, with basis  $\mathbf{v}_1$ , so there is one exponential solution,  $\mathbf{x}_1(t) = e^{\lambda_1 t} \mathbf{v}_1$ .
- $\lambda_2 = -1$  : Eigenspace has dimension 2 with basis  $\mathbf{v}_2$  and  $\mathbf{v}_3$ , so there are two linearly independent exponential solutions  $\mathbf{x}_2(t) = e^{\lambda_2 t} \mathbf{v}_2$  and  $\mathbf{x}_3(t) = e^{\lambda_2 t} \mathbf{v}_3$ .
- $\mathbf{x}_1$ ,  $\mathbf{x}_2$ , and  $\mathbf{x}_3$  are a fundamental set of solutions.

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## Repeated Eigenvalues – Example 2

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -4 & 1 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 3)(\lambda + 1)^2$
- $\lambda_1 = -3$  : Eigenspace has dimension 1. There is one exponential solution  $\mathbf{x}_1(t) = e^{-3t}(-1/2, 3/2, 1)^T$ .
- $\lambda_2 = -1$  : Eigenspace has dimension 1. There is only one exponential solution  $\mathbf{x}_2(t) = e^{-t}(-1/2, 1, 1)^T$ .
- We need a third solution. We need a new idea.

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## Multiplicities

$A$  an  $n \times n$  matrix

- Distinct eigenvalues  $\lambda_1, \dots, \lambda_k$ .
- The characteristic polynomial factors as

$$p(\lambda) = (\lambda - \lambda_1)^{q_1} (\lambda - \lambda_2)^{q_2} \dots (\lambda - \lambda_k)^{q_k}.$$

- The *algebraic multiplicity* of  $\lambda_j$  is  $q_j$ .
- The *geometric multiplicity* of  $\lambda_j$  is  $d_j$ , the dimension of the eigenspace of  $\lambda_j$ .

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- We always have:
  - ♦  $q_1 + q_2 + \dots + q_k = n$ .
  - ♦  $1 \leq d_j \leq q_j$ .
  - ♦ There are  $d_j$  linearly independent exponential solutions corresponding to  $\lambda_j$ .
  - ♦ If  $d_j = q_j$  for all  $j$  we have  $n$  linearly independent solutions.
- If  $d_j < q_j$  we have trouble.

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## New Approach

- $D = 1 : x' = ax$ 
  - ♦ Solution  $x(t) = Ce^{at}$ .
- $D > 1 : \mathbf{x}' = A\mathbf{x}$ 
  - ♦ Tried  $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ .
    - ▶ Worked well except when eigenvalues have multiplicity greater than 1.
  - ♦ Why not  $\mathbf{x}(t) = e^{tA}\mathbf{v}$ ?
- But what is  $e^{tA}$ ?

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## Exponential of a Matrix

**Definition:** The *exponential* of the  $n \times n$  matrix  $A$  is the  $n \times n$  matrix

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots \\ = \sum_0^{\infty} \frac{1}{n!}A^n.$$

- **Examples:**

- $A = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \Rightarrow e^A = \begin{pmatrix} e^{r_1} & 0 \\ 0 & e^{r_2} \end{pmatrix}.$
- $e^{\lambda I} = e^{\lambda}I. \quad e^{0I} = I.$

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## Properties

- $A$  commutes with  $e^A$ ,

$$Ae^A = e^A A.$$

- If  $A$  and  $B$  commute (i.e.,  $AB = BA$ ), then

$$e^{A+B} = e^A \cdot e^B.$$

- The inverse of  $e^A$  is  $e^{-A}$ .

- $\frac{d}{dt}e^{tA} = Ae^{tA}.$

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## A Very Important Fact

**Theorem:** The solution to the initial value problem

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad \mathbf{x}(0) = \mathbf{v}$$

is given by  $\mathbf{x}(t) = e^{tA}\mathbf{v}.$

- However computing  $e^{tA}$  is not easy.

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$\mathbf{x}'$

$\mathbf{x}(0)$

## Key to Computing $e^{tA}$ or $e^{tA}\mathbf{v}$

Suppose that  $A$  an  $n \times n$  matrix, and  $\lambda$  a number (an eigenvalue).

- Then  $A = \lambda I + (A - \lambda I)$ , and  $\lambda I$  &  $A - \lambda I$  commute. Therefore

$$\begin{aligned} e^{tA} &= e^{t[\lambda I + (A - \lambda I)]} \\ &= e^{t\lambda I} \cdot e^{t(A - \lambda I)} \\ &= e^{\lambda t} \cdot e^{t(A - \lambda I)} \\ &= e^{\lambda t} \cdot [I + t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \dots] \end{aligned}$$

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 $e^{tA}$ 

Solution

## $e^{tA}\mathbf{v}$ , $\mathbf{v}$ an Eigenvector

Let  $\lambda$  be an eigenvalue and  $\mathbf{v}$  an associated eigenvector.

Then  $(A - \lambda I)\mathbf{v} = \mathbf{0}$ , so

$$\begin{aligned} e^{tA}\mathbf{v} &= e^{\lambda t} \cdot e^{t(A - \lambda I)}\mathbf{v} \\ &= e^{\lambda t} [I + t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \dots]\mathbf{v} \\ &= e^{\lambda t} [\mathbf{v} + t(A - \lambda I)\mathbf{v} + \frac{t^2}{2!}(A - \lambda I)^2\mathbf{v} + \dots] \\ &= e^{\lambda t}\mathbf{v} \end{aligned}$$

- The infinite series truncates, so we can compute  $e^{tA}\mathbf{v}$ .

Return

Key

Solution

## Matrices with One Eigenvalue

$A$  has characteristic polynomial  $p(\lambda) = (\lambda - \lambda_1)^n$ .

- *Cayley-Hamilton Theorem*: If  $p(\lambda)$  is the characteristic polynomial of the matrix  $A$  then  $p(A) = 0I$ .
- In our case  $(A - \lambda_1 I)^n = 0I$ , so

$$\begin{aligned} e^{tA} &= e^{\lambda_1 t} \cdot [I + t(A - \lambda_1 I) + \frac{t^2}{2!}(A - \lambda_1 I)^2 \\ &\quad + \dots + \frac{t^{n-1}}{(n-1)!}(A - \lambda_1 I)^{n-1}] \end{aligned}$$

Return

Key

Eigenvector

### Example 3

$$A = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 2)^2$ .

$$A + 2I = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \quad (A + 2I)^2 = 0I$$

$$\begin{aligned} e^{tA} &= e^{-2t}[I + t(A + 2I)] \\ &= e^{-2t} \begin{pmatrix} 1-t & t \\ -t & 1+t \end{pmatrix}. \end{aligned}$$

Return

1 eigenvalue

### Example 4

$$A = \begin{pmatrix} 0 & -9 & 27 \\ -2 & 3 & -18 \\ -1 & 3 & -12 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 3)^3$ .  $(A + 3I)^2 = 0I$ .

$$\begin{aligned} e^{tA} &= e^{-3t}[I + t(A + 3I)] \\ &= e^{-3t} \begin{pmatrix} 1+3t & -9t & 27t \\ -2t & 1+6t & -18t \\ -t & 3t & 1-9t \end{pmatrix}. \end{aligned}$$

Return

1 eigenvalue

### Example 2, Reprise

- Distinct eigenvalues  $\lambda_1 = -3$  &  $\lambda_2 = -1$
- Different from previous two examples.
- $\lambda_1 = -3$  has algebraic multiplicity 1, and geometric multiplicity 1. So there is one exponential solution

$$\mathbf{x}_1(t) = e^{\lambda_1 t} \mathbf{v}_1 = e^{-3t} (-1/2, 3/2, 1)^T.$$

- $\lambda_2 = -1$  has algebraic multiplicity 2, and geometric multiplicity 1. So there is only one exponential solution

$$\mathbf{x}_2(t) = e^{\lambda_2 t} \mathbf{v}_2 = e^{-t} (-1/2, 1, 1)^T.$$

Return

Example 2

Multiplicities

- However,  $\text{null}((A - \lambda_2 I)^2)$  has dimension 2, with basis  $(0, 1, 1)^T$  and  $(1, 0, 0)^T$ . If  $\mathbf{v} \in \text{null}((A - \lambda_2 I)^2)$  then

$$\begin{aligned} e^{tA}\mathbf{v} &= e^{\lambda_2 t} \left[ I + t(A - \lambda_2 I) + \frac{t^2}{2!}(A - \lambda_2 I)^2 + \cdots \right] \mathbf{v} \\ &= e^{\lambda_2 t} [\mathbf{v} + t(A - \lambda_2 I)\mathbf{v}]. \end{aligned}$$

- ♦  $\mathbf{v}_2$  is in  $\text{null}((A - \lambda_2 I)^2)$
- Using  $\mathbf{v}_3 = (1, 0, 0)^T$  we get the third solution

$$\begin{aligned} \mathbf{x}_3(t) &= e^{tA}\mathbf{v}_3 = e^{-t}[\mathbf{v}_3 + t(A + I)\mathbf{v}_3] \\ &= e^{-t}(1 + 2t, -4t, -4t)^T. \end{aligned}$$

- $\mathbf{x}_1$ ,  $\mathbf{x}_2$ , and  $\mathbf{x}_3$  are a fundamental set of solutions.

Return      Key       $\mathbf{x}_1$       Example 2      Multiplicities

### Summary

- In Examples 3 & 4 the matrix has one eigenvalue.
  - ♦ The series for  $e^{t(A-\lambda I)}$  truncates to a finite sum.
- In Example 2 the matrix had two eigenvalues.
  - ♦ The series for  $e^{t(A-\lambda I)}$  does not truncate for any  $\lambda$ .
  - ♦ However, the series for  $e^{t(A-\lambda_2 I)}\mathbf{v}$  does truncate if  $(A - \lambda_2 I)^2\mathbf{v} = \mathbf{0}$ .

Return      Key

### Generalized Eigenvectors

**Definition:** If  $\lambda$  is an eigenvalue of  $A$  and  $(A - \lambda I)^p\mathbf{v} = \mathbf{0}$  for some integer  $p \geq 1$ , then  $\mathbf{v}$  is called a *generalized eigenvector* associated with  $\lambda$ .

- The series for  $e^{t(A-\lambda I)}\mathbf{v}$  truncates to a finite sum if  $\mathbf{v}$  is a generalized eigenvector associated with  $\lambda$ .
- We can compute  $e^{tA}\mathbf{v}$ .

Return

**Theorem:** If  $\lambda$  is an eigenvalue of  $A$  with algebraic multiplicity  $q$ , then there is an integer  $p \leq q$  such that  $\text{null}((A - \lambda I)^p)$  has dimension  $q$ .

- For each generalized eigenvector  $\mathbf{v}$  we can compute  $e^{tA}\mathbf{v}$ .
- We can find  $q$  linearly independent solutions associated with the eigenvalue  $\lambda$ .

Return

Key

### Procedure for $\lambda$ of algebraic multiplicity $q$

To find  $q$  linearly independent solutions associated with  $\lambda$ :

- Find the smallest integer  $p$  such that  $\text{null}((A - \lambda I)^p)$  has dimension  $q$ .
- Find a basis  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_q$  of  $\text{null}((A - \lambda I)^p)$ .
- For  $j = 1, 2, \dots, q$

$$\begin{aligned} \mathbf{x}_j(t) &= e^{tA}\mathbf{v}_j \\ &= e^{\lambda t}[\mathbf{v}_j + t(A - \lambda I)\mathbf{v}_j + \frac{t^2}{2!}(A - \lambda I)^2\mathbf{v}_j \\ &\quad + \dots + \frac{t^{p-1}}{(p-1)!}(A - \lambda I)^{p-1}\mathbf{v}_j] \end{aligned}$$

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### Example

- Use MATLAB.

### Procedure for a Complex Eigenvalue

If  $\lambda$  is complex of algebraic multiplicity  $q$ . Then  $\bar{\lambda}$  also has multiplicity  $q$ .

- Find the smallest integer  $p$  such that  $\text{null}((A - \lambda I)^p)$  has dimension  $q$ .
- Find a basis  $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_q$  of  $\text{null}((A - \lambda I)^p)$ .
- For  $j = 1, 2, \dots, q$

$$\mathbf{z}_j(t) = e^{tA}\mathbf{w}_j.$$

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- For  $j = 1, 2, \dots, q$

$$\begin{aligned} \mathbf{z}_j(t) = e^{\lambda t} & [\mathbf{w}_j + t(A - \lambda I)\mathbf{w}_j \\ & + \frac{t^2}{2!}(A - \lambda I)^2\mathbf{w}_j + \dots \\ & + \frac{t^{p-1}}{(p-1)!}(A - \lambda I)^{p-1}\mathbf{w}_j] \end{aligned}$$

- For  $j = 1, 2, \dots, q$  set

$$\mathbf{x}_j(t) = \text{Re}(\mathbf{z}_j(t)) \quad \text{and}$$

$$\mathbf{y}_j(t) = \text{Im}(\mathbf{z}_j(t)).$$