Math 211
Lecture #4
Separable Equations

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Autonomous Equations

- General equation: \( \frac{dy}{dt} = f(t, y) \)
- Autonomous equation: \( \frac{dy}{dt} = f(y) \)

Examples:
- \( \frac{dy}{dt} = t - y^2 \) is not autonomous.
- \( \frac{dy}{dt} = y(1 - y) \) is autonomous.

In an autonomous equation the right-hand side has no explicit dependence on the independent variable.

Equilibrium Points

- An equilibrium point for the autonomous equation \( \frac{dy}{dt} = f(y) \) is a point \( y_0 \) such that \( f(y_0) = 0 \).
- Corresponding to the equilibrium point \( y_0 \), there is the constant equilibrium solution \( y(t) = y_0 \).
- Example: \( \frac{dy}{dt} = y(2 - y)/3 \) is an autonomous equation.
  - The equilibrium points are \( y_1 = 0 \) and \( y_2 = 2 \).
  - The corresponding equilibrium solutions are \( y_1(t) = 0 \) and \( y_2(t) = 2 \).
Between Equilibrium Points

- The graphs of solutions to first order equations cannot cross (uniqueness theorem).
- \( \frac{dy}{dt} = f(y) > 0 \Rightarrow y(t) \) is increasing.
- \( \frac{dy}{dt} = f(y) < 0 \Rightarrow y(t) \) is decreasing.
- Example: \( \frac{dy}{dt} = y(2-y)/3 \)

Separable Equations

- General differential equation: \( \frac{dy}{dt} = f(t, y) \)
- Separable differential equation: \( \frac{dy}{dt} = g(y)h(t) \)
- In a separable equation the right-hand side is a product of a function \( h(t) \) of the independent variable \( t \) and a function \( g(y) \) of the unknown function \( y \).
- Examples:
  - \( \frac{dy}{dt} = t - y^2 \) is not separable.
  - \( \frac{dy}{dt} = t \sec y \) is separable.
  - Any autonomous equation \( y' = f(y) \) is separable.

Solving Separable Equations

Example: \( y' = \frac{dy}{dt} = t \sec y \)

- Step 1: Separate the variables:
  \[
  \frac{dy}{\sec y} = t \ dt \quad \text{or} \quad \cos y \ dy = t \ dt
  \]
- We have to worry about dividing by 0, but in this case \( \sec y \) is never equal to 0.
• Step 2: Integrate both sides of \( \cos y \, dy = t \, dt \)

\[
\int \cos y \, dy = \int t \, dt
\]

\[
\sin y + C_1 = \frac{1}{2} t^2 + C_2 \quad \text{or} \quad \sin(y(t)) = \frac{1}{2} t^2 + C
\]

where \( C = C_2 - C_1 \).

• Step 3: Solve \( \sin(y(t)) = \frac{1}{2} t^2 + C \) for \( y(t) \)

- We get

\[
y(t) = \arcsin \left( C + \frac{1}{2} t^2 \right).
\]

- This is the general solution to \( \frac{dy}{dt} = t \sec y \).

**Solving Separable Equations**

\[
\frac{dy}{dt} = g(y)h(t)
\]

The three step solution process:

1. Separate the variables. \( \frac{dy}{g(y)} = h(t) \, dt \) if \( g(y) \neq 0 \).
2. Integrate both sides. \( \int \frac{dy}{g(y)} = \int h(t) \, dt \)
3. Solve for \( y(t) \).
Examples

- $y' = ry$ with $y(0) = -2, 0, 3$
- $y' = 2ty$ with $y(0) = -1, 0, 2$
- $R' = \frac{\sin t}{1 + R}$ with $R(0) = 1, -2, -1$
- $x' = \frac{3x^2}{1 + 2x^2}$ with $x(0) = 1, 0$
- $y' = 1 + y^2$ with $y(0) = -1, 0, 1$

Why the Method Works

\[
\frac{dy}{dt} = g(y)h(t)
\]

\[
\frac{1}{g(y)} \frac{dy}{dt} = h(t) \quad \text{if } g(y) \neq 0
\]

\[
\int \frac{1}{g(y)} \frac{dy}{dt} \, dt = \int h(t) \, dt
\]

\[
\int \frac{1}{g(y)} \, dy = \int h(t) \, dt
\]