# Math 211

Lecture #4

Models of Motion

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# The Modeling Process

- It is based on experiment and/or observation.
- It is iterative.
  - For motion we have $\geq 6$ iterations.
  - After each change in the model it must be checked by experimentation and observation.
- It is rare that a model captures all aspects of the phenomenon.

# Linear Motion

- Motion in one dimension
  - Example – motion of a ball in the earth's gravity.
- $x(t)$ is the distance from a reference position.
  - $x(t)$ is the height of the ball above the surface of the earth.
- Velocity: $v = x'$
- Acceleration: $a = v' = x''$.

Return

- Acceleration due to gravity is (approximately) constant near the surface of the earth

$$F = -mg \quad g = 9.8m/s^2$$

- Newton's second law: $F = ma$
- Equation of motion: $ma = -mg$, which becomes

$$x'' = -g \quad \text{or} \quad \begin{aligned} x' &= v, \\ v' &= -g. \end{aligned}$$

- Solving the system
$$x' = v,$$
$$v' = -g$$

- Integrate the second equation:

$$v(t) = -gt + c_1$$

- Integrate the first equation:

$$x(t) = -\frac{1}{2}gt^2 + c_1 t + c_2.$$

# Resistance of the Medium

- Force of resistance

$$R(x,v) = -r(x,v)v \quad \text{where} \quad r(x,v) \geq 0.$$

- Different models
  - ◇ Resistance proportional to velocity.

    $$R(x,v) = -rv, \quad r \text{ a constant.}$$

  - ◇ Magnitude of resistance proportional to the square of the velocity.

    $$R(x,v) = -k|v|v.$$

# $R(x, v) = -rv$

- Total force: $F = -mg - rv$
- Newton's second law: $F = ma$
- Equation of motion:

$$mx'' = -mg - rv \quad \text{or} \quad \begin{aligned} x' &= v, \\ v' &= -\frac{mg + rv}{m}. \end{aligned}$$

Return

# $R(x, v) = -rv$ (cont.)

- The equation $v' = -\frac{mg + rv}{m}$ for $v$ is separable.
- Solution is $v(t) = Ce^{-rt/m} - \frac{mg}{r}$.
- Notice
$$\lim_{t\to\infty} v(t) = -\frac{mg}{r}.$$
- The *terminal velocity* is $v_{\text{term}} = -\frac{mg}{r}$.

$$R(x, v) = -k|v|v$$

- Total force: $F = -mg - k|v|v$.
- Equation of motion:

$$mx'' = -mg - k|v|v \quad \text{or} \quad \begin{aligned} x' &= v, \\ v' &= -g - \frac{k|v|v}{m}. \end{aligned}$$

- The equation for $v$ is separable.

- Suppose a ball is dropped from a high point. Then $v < 0$.
- The equation is

$$\begin{aligned} v' &= \frac{-mg + kv^2}{m} \\ &= -\frac{k}{m}\left[\frac{mg}{k} - v^2\right] \\ &= -\frac{k}{m}\left[\alpha^2 - v^2\right], \quad \text{where } \alpha = \sqrt{mg/k}. \end{aligned}$$

- The solution is

$$v(t) = \sqrt{\frac{mg}{k}}\,\frac{Ae^{-2t\sqrt{kg/m}} - 1}{Ae^{-2t\sqrt{kg/m}} + 1}.$$

- The terminal velocity is

$$v_{\text{term}} = -\sqrt{mg/k}.$$