Math 211

Lecture #8

Qualtitative Analysis

February 2, 2001

Qualitative Analysis

- Ways to discover the properties of solutions without solving the equation.
- Works best with autonomous equations

$$y' = f(y)$$

- Example: $y' = \sin y$

Properties of Autonomous Equations

- The direction field does not depend on t
- Solution curves can be translated left and right to get other solution curves.
  - If $y(t)$ is a solution, so is $y_1 = y(t + c)$ for any constant c.

Equilibrium Points & Solutions

$$y' = f(y) \qquad y' = \sin y$$

- Equilibrium point: $f(y_0) = 0$.
- Equilibrium solution: $y(t) = y_0$.
- $\sin y = 0 \quad \Leftrightarrow \quad y = k\pi, \quad k = 0, \pm 1, \dots$
- $y' = \sin y$ has infinitely many equilibrium solutions:
  - ◇ $y_k(t) = k\pi$ for $k = 0, \pm 1, \pm 2, \dots$

Return

Between the Equilibrium Points

$$
\begin{aligned}
0 < y < \pi \;&\Rightarrow\; \sin y > 0 \\
&\Rightarrow\; y'(t) = \sin y(t) > 0 \\
&\Rightarrow\; y(t) \text{ is increasing}
\end{aligned}
$$

- By uniqueness, $0 < y(t) < \pi$ for all t.
- Thus $y(t) \nearrow \pi$ as $t \to \infty$
  and $y(t) \searrow 0$ as $t \to -\infty$

Between the Equilibrium Points

$$
\begin{aligned}
-\pi < y < 0 &\Rightarrow \sin y < 0 \\
&\Rightarrow y'(t) = \sin y(t) < 0 \\
&\Rightarrow y(t) \text{ is decreasing}
\end{aligned}
$$

- By uniqueness, $0 > y(t) > -\pi$ for all t.
- Thus $y(t) \searrow -\pi \quad$ as $t \to \infty$
  and $y(t) \nearrow 0 \quad$ as $t \to -\infty$

Stable & Unstable EPs

An equilibrium point y_0 is

- *asymptotically stable* if all solutions starting near y_0 converge to y_0 as $t \to \infty$.
- *unstable* if there are solutions starting arbitrarily close to y_0 which move away from y_0 as t increases.
- There are 4 possibilities:

The Phase Line for $y' = f(y)$

- The phase line is a y-axis, showing
  - the equilibrium points and
  - the direction of the flow between the equilibrium points.
- The y-axis in the plot of $y \longrightarrow f(y)$.
- The y-axis in the ty-plane where solutions are plotted.

Terminal Velocity

- Magnitude of the resistance proportional to the square of the velocity:

$$v' = -g - k|v|v/m$$

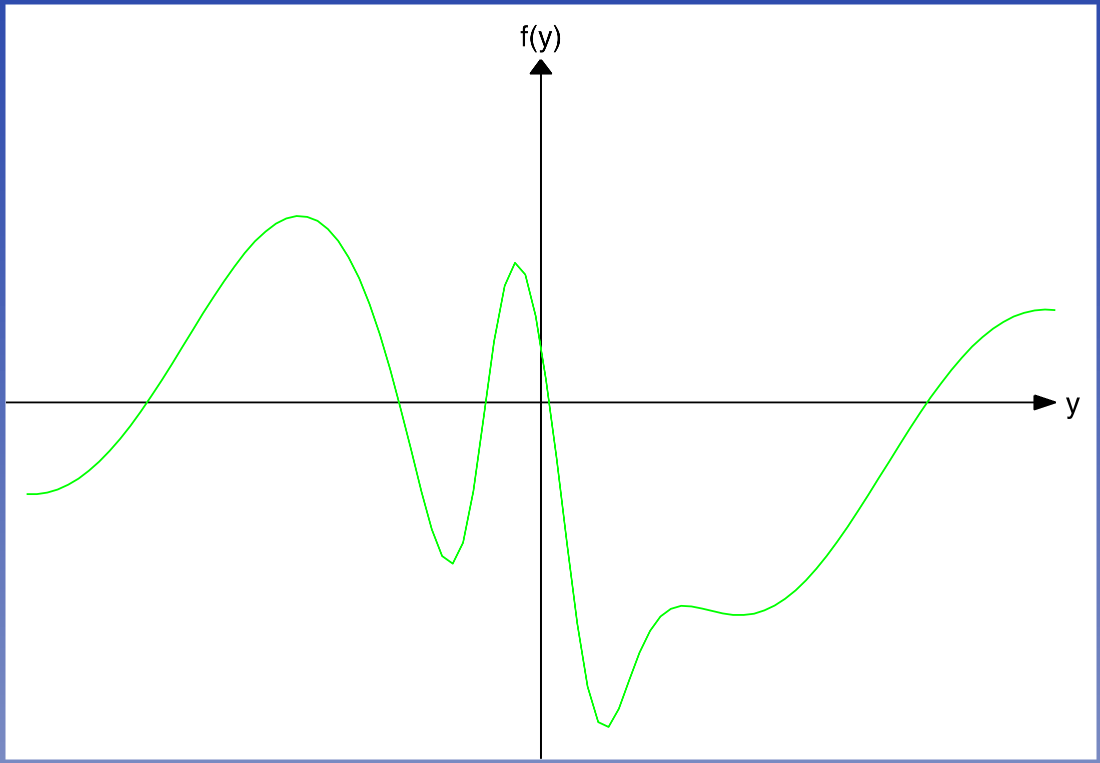
- One equilibrium point at

$$v_{\text{term}} = -\sqrt{\frac{mg}{k}}.$$

- v_{term} is asymptotically stable.

Qualitative Analysis of $y' = f(y)$.

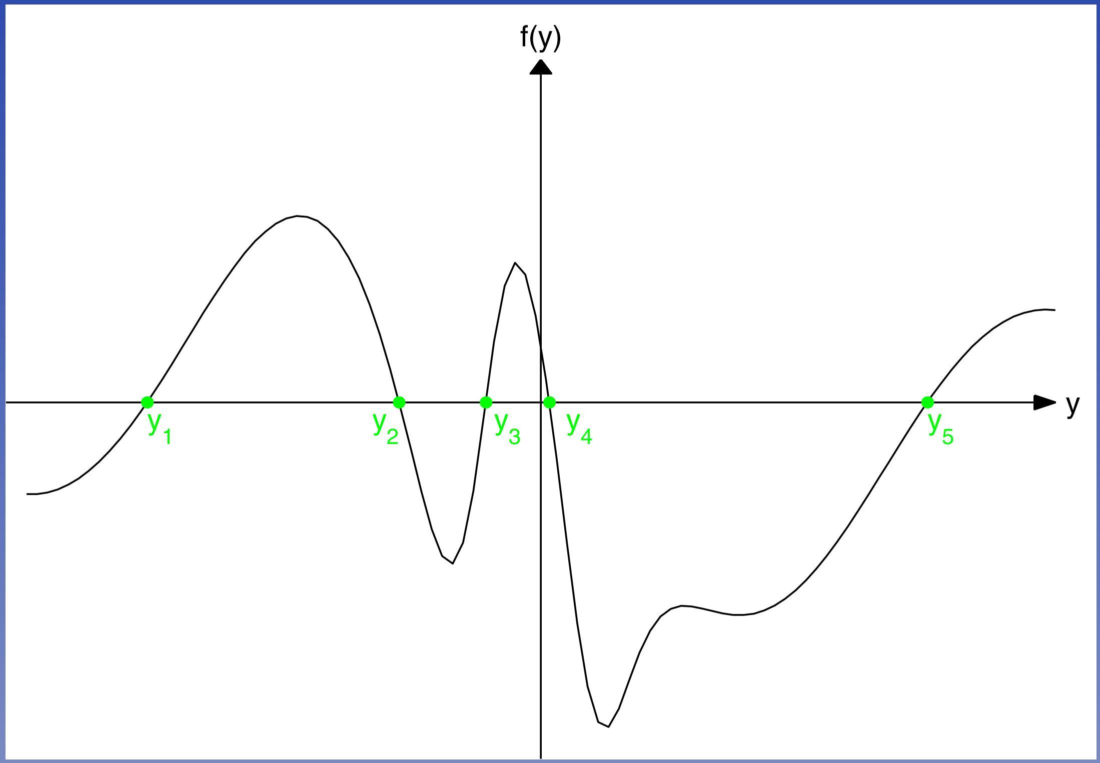
1. Graph $y \to f(y)$.



Return

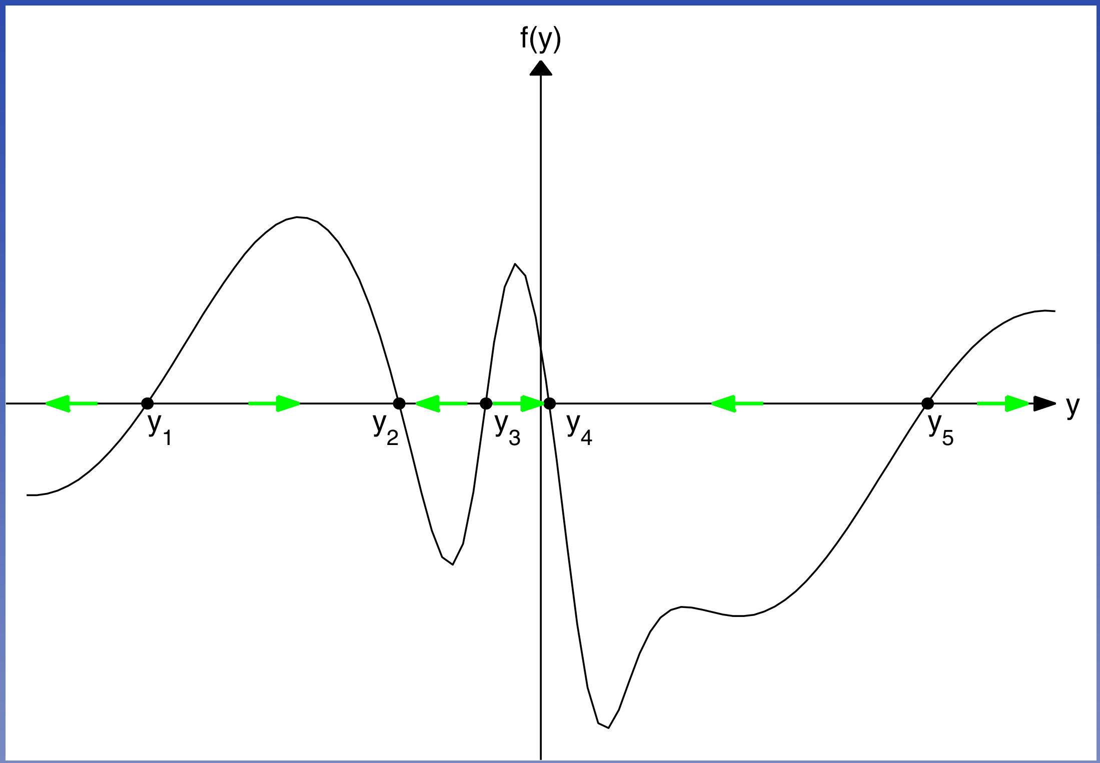
Qualitative Analysis of $y' = f(y)$.

2. Find the equilibrium points where $f(y) = 0$.



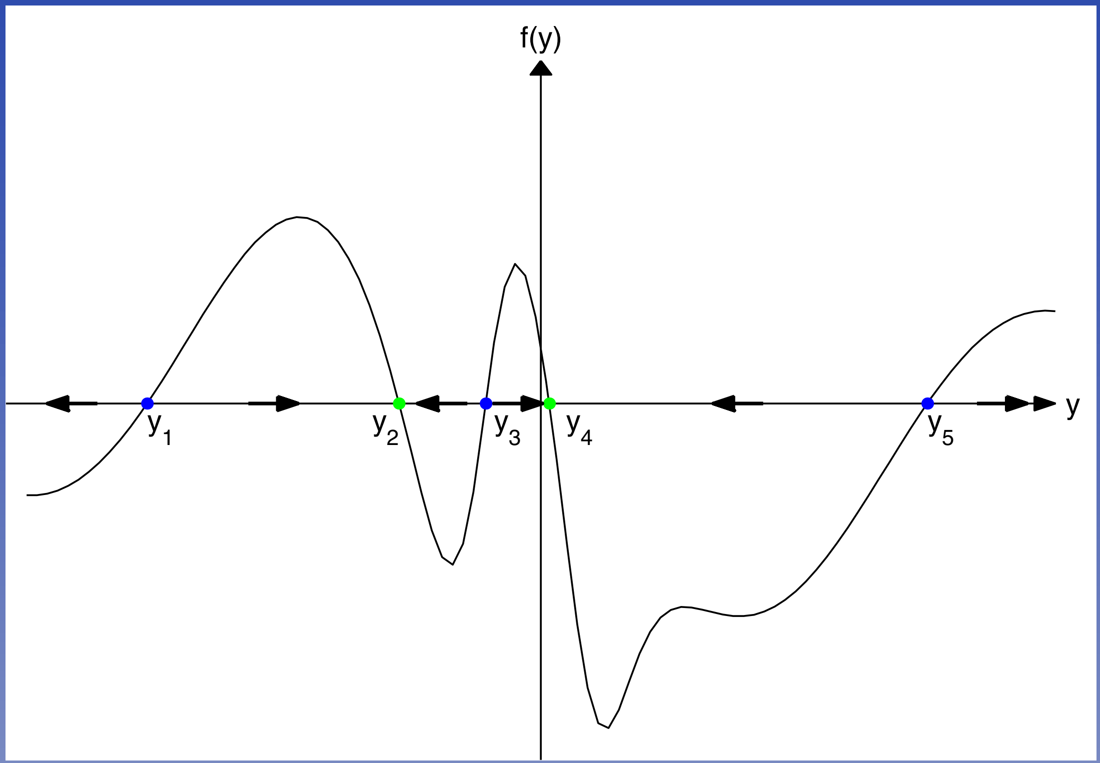
Qualitative Analysis of $y' = f(y)$.

3. Determine the behavior between eq. pts.



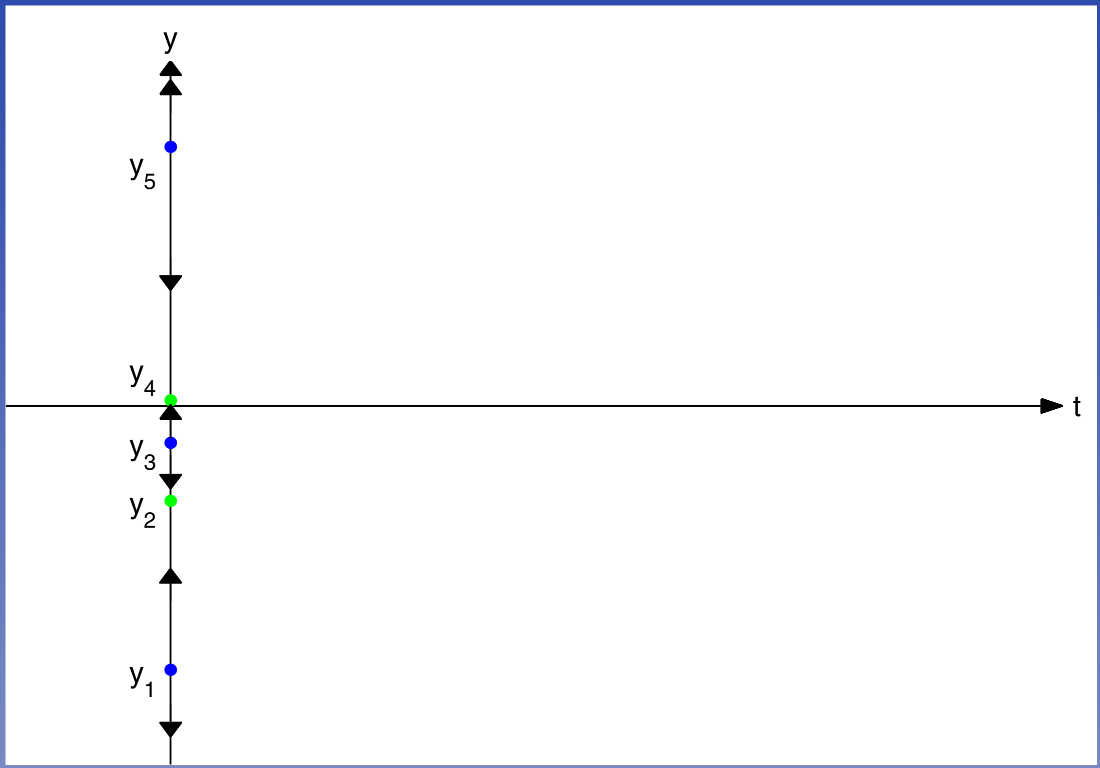
Qualitative Analysis of $y' = f(y)$.

4. Analyze the equilibrium points.



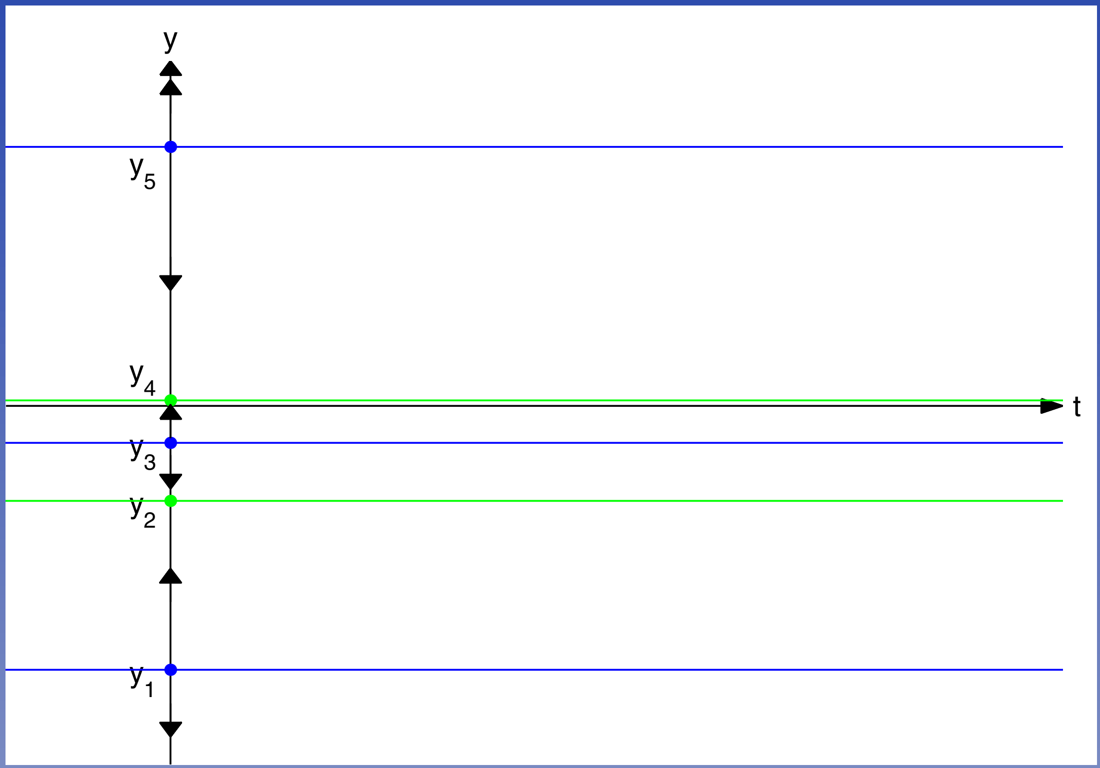
Qualitative Analysis of $y' = f(y)$.

5. Transfer the phase line to ty-space.



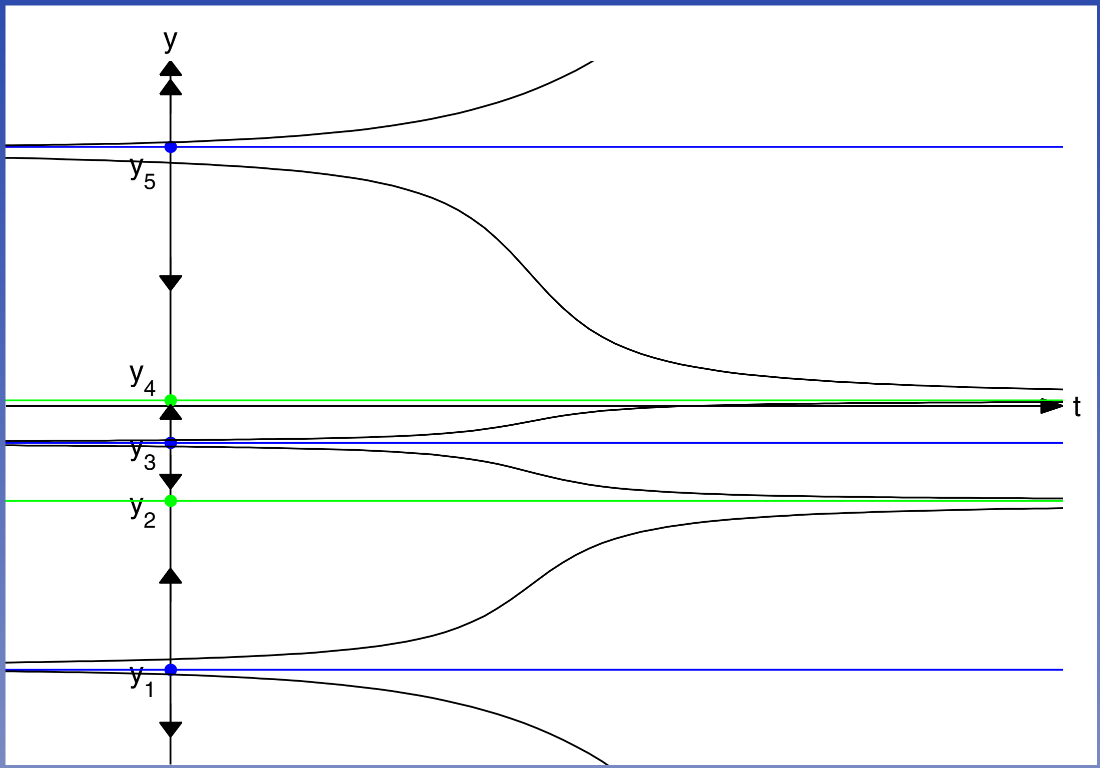
Qualitative Analysis of $y' = f(y)$.

6. Plot the equilibrium solutions.



Qualitative Analysis of $y' = f(y)$.

7. Plot other solutions approximately.



Modeling Population

- Assume population changes due to births and deaths.
- Births are roughly proportional to population

$$B = bP \quad b \text{ is the } \textit{birth rate}$$

- Deaths are roughly proportional to population

$$D = dP \quad d \text{ is the } \textit{death rate}$$

Modeling Population

- Rate of change = births − deaths

$$\frac{dP}{dt} = B - D$$
$$= bP - dP$$
$$= (b - d)P$$
$$= rP$$

- $r = b - d$ is the *reproductive rate*.

The Malthusian Model

- b and d not necessarily constants
  - ◇ Can depend on P, and perhaps also on t.
- If there exist sufficient resources (nutrients and space), b and d will be almost constant. Then the reproductive rate $r = b - d$ is also a constant.
- This is the *Malthusian model*.

The Malthusian Model

$$\frac{dP}{dt} = rP \quad \text{with} \quad P(0) = P_0$$

- Solution: $P(t) = P_0 e^{rt}$
  - ◇ If $r = b - d > 0$, $P(t)$ grows exponentially.
  - ◇ If $r = b - d < 0$, $P(t)$ decays exponentially.

The Malthusian Model

Under what circumstances could the Malthusian model be a good model?

- Requires unlimited resources.
- Laboratory experiments with small populations.
- Populations always outgrow the Malthusian model. This was the point that was made by Malthus.