

Math 211

Lecture #32

Higher Order Equations
Harmonic Motion

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Solutions to $y'' + py' + qy \equiv 0$.

- Equivalent system: $\mathbf{x}' = A\mathbf{x}$, where

$$\mathbf{x} = \begin{pmatrix} y \\ y' \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}.$$

- Look for exponential solutions $y(t) = e^{\lambda t}$.
- *Characteristic equation:* $\lambda^2 + p\lambda + q = 0$.
- *Characteristic polynomial:* $\lambda^2 + p\lambda + q$.
- Same for the 2nd order equation and the system.
- Need two linearly independent solutions.

Real Roots

- If λ is a root to the **characteristic polynomial** then $y(t) = e^{\lambda t}$ is a solution.
 - ◇ $y'' + 4y' + 3y = 0$
- If λ is a root to the characteristic polynomial of multiplicity 2, then $y_1(t) = e^{\lambda t}$ and $y_2(t) = te^{\lambda t}$ are linearly independent solutions.
 - ◇ $y'' + 4y' + 4y = 0$

Complex Roots

- If $\lambda = \alpha + i\beta$ is a complex root of the **characteristic equation**, then so is $\bar{\lambda} = \alpha - i\beta$.
- A complex valued fundamental set of solutions is

$$z(t) = e^{\lambda t} \quad \text{and} \quad \bar{z}(t) = e^{\bar{\lambda}t}.$$

- A real valued fundamental set of solutions is

$$x(t) = e^{\alpha t} \cos \beta t \quad \text{and} \quad y(t) = e^{\alpha t} \sin \beta t.$$

$$\diamond y'' + 4y' + 8y = 0$$

Examples

- $y'' - 5y' + 6y = 0.$
- $y'' + 25y = 0.$
- $y'' + 4y' + 13y = 0.$

The Vibrating Spring

Newton's second law: $ma = \text{total force}$.

- Forces acting:
 - ◇ Gravity mg .
 - ◇ Restoring force $R(x)$.
 - ◇ Damping force $D(v)$.
 - ◇ External force $F(t)$.

- Newton's law becomes

$$ma = mg + R(x) + D(v) + F(t)$$

- Hooke's law: $R(x) = -kx$. $k > 0$ is the *spring constant*.

- Spring-mass equilibrium $x_0 = mg/k$.

- Set $y = x - x_0$. Newton's law becomes

$$my'' = -ky + D(y') + F(t).$$

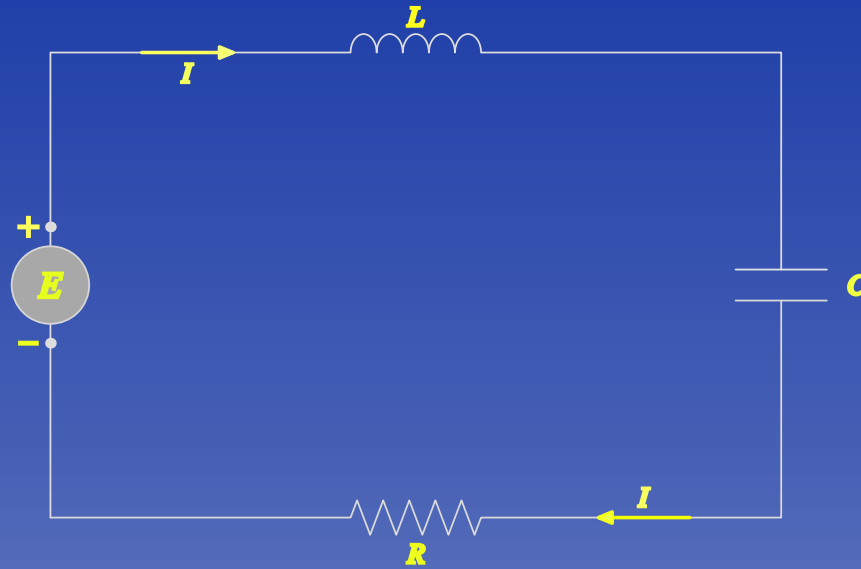
- Damping force $D(y') = -\mu y'$.
- Newton's law becomes

$$my'' = -ky - \mu y' + F(t), \quad \text{or}$$

$$my'' + \mu y' + ky = F(t), \quad \text{or}$$

$$y'' + \frac{\mu}{m}y' + \frac{k}{m}y = \frac{1}{m}F(t).$$

RLC Circuit



$$LI'' + RI' + \frac{1}{C}I = E'(t), \quad \text{or}$$
$$I'' + \frac{R}{L}I' + \frac{1}{LC}I = \frac{1}{L}E'(t).$$

Harmonic Motion

- Spring: $y'' + \frac{\mu}{m}y' + \frac{k}{m}y = \frac{1}{m}F(t)$.
- Circuit: $I'' + \frac{R}{L}I' + \frac{1}{LC}I = \frac{1}{L}E'(t)$.
- Essentially the same equation. Use

$$x'' + 2cx' + \omega_0^2x = f(t).$$

- The equation for *harmonic motion*.

$$x'' + 2cx' + \omega_0^2 x = f(t).$$

- ω_0 is the *natural frequency*.
 - ◇ Spring: $\omega_0 = \sqrt{k/m}$.
 - ◇ Circuit: $\omega_0 = \sqrt{1/LC}$.
- c is the *damping constant*.
- $f(t)$ is the *forcing term*.

Simple Harmonic Motion

No **forcing** , and no damping.

$$x'' + \omega_0^2 x = 0$$

- $p(\lambda) = \lambda^2 + \omega_0^2$, $\lambda = \pm i\omega_0$.

Fundamental set of **solutions**

$$x_1(t) = \cos \omega_0 t \quad \& \quad x_2(t) = \sin \omega_0 t.$$

General solution

$$x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t.$$

- Every solution is periodic with **frequency** ω_0 .
 - ◇ ω_0 is the natural frequency.
 - ◇ The period is $T = 2\pi/\omega_0$.

Amplitude and Phase

- Put C_1 and C_2 in polar coordinates:

$$C_1 = A \cos \phi, \text{ \& } C_2 = A \sin \phi.$$

- Then $x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$

$$= A \cos(\omega_0 t - \phi).$$

- A is the *amplitude*; $A = \sqrt{C_1^2 + C_2^2}$.
- ϕ is the *phase*; $\tan \phi = C_2/C_1$.

Examples

- $C_1 = 3, C_2 = 4 \Rightarrow A = 5, \phi = 0.9273.$
- $C_1 = -3, C_2 = 4 \Rightarrow A = 5, \phi = 2.2143.$
- $C_1 = -3, C_2 = -4 \Rightarrow A = 5, \phi = -2.2143.$

Example

$$x'' + 16x = 0, x(0) = -2 \text{ \& } x'(0) = 4$$

- **Natural frequency:** $\omega_0^2 = 16 \Rightarrow \omega_0 = 4$.
- **General solution:** $x(t) = C_1 \cos 4t + C_2 \sin 4t$.
- **IC:** $-2 = x(0) = C_1$, and $4 = x'(0) = 4C_2$.
- **Solution**

$$\begin{aligned} x(t) &= -2 \cos 2t + \sin 2t \\ &= \sqrt{5} \cos(2t - 2.6779). \end{aligned}$$

Damped Harmonic Motion

$$x'' + 2cx' + \omega_0^2 x = 0$$

- $p(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2$; roots $-c \pm \sqrt{c^2 - \omega_0^2}$.
- Three cases
 - ◇ $c < \omega_0$ *Underdamped*
 - ◇ $c > \omega_0$ *Overdamped*
 - ◇ $c = \omega_0$ *Critically damped*

Underdamped

- $c < \omega_0$
- Two **complex** roots λ and $\bar{\lambda}$, where $\lambda = -c + i\omega$ and $\omega = \sqrt{\omega_0^2 - c^2}$.
- General solution

$$\begin{aligned}x(t) &= e^{-ct} [C_1 \cos \omega t + C_2 \sin \omega t] \\ &= Ae^{-ct} \cos(\omega t - \phi)\end{aligned}$$

Overdamped

- $c > \omega_0$, so two real roots

$$\lambda_1 = -c - \sqrt{c^2 - \omega_0^2}$$

$$\lambda_2 = -c + \sqrt{c^2 - \omega_0^2}.$$

- $\lambda_1 < \lambda_2 < 0$.
- General solution

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

Critically Damped

- $c = \omega_0$
- One negative **real root** $\lambda = -c$ with multiplicity 2.
- General solution

$$x(t) = e^{-ct}[C_1 + C_2t].$$