Math 211

Lecture #35

Forced Harmonic Motion

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Forced Harmonic Motion

Assume an oscillatory forcing term:

$$y'' + 2cy' + \omega_0^2 y = A\cos\omega t$$

- A is the forcing amplitude
- ω is the forcing frequency
- ω_0 is the natural frequency.
- c is the damping constant.

Forced Undamped Motion

$$y'' + \omega_0^2 y = A\cos\omega t$$

- Homogeneous equation: $y'' + \omega_0^2 y = 0$
  - ◇ General solution

$$y(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t.$$

- $\omega \neq \omega_0$: Particular solution

$$x_p(t) = \frac{A}{\omega_0^2 - \omega^2}\cos\omega t.$$

- $\omega \neq \omega_0$
  - ◇ Initial conditions $x(0) = x'(0) = 0 \Rightarrow$

$$x(t) = \frac{A}{\omega_0^2 - \omega^2}[\cos \omega t - \cos \omega_0 t].$$

  - ◇ Set $\overline{\omega} = \dfrac{\omega_0 + \omega}{2}$ and $\delta = \dfrac{\omega_0 - \omega}{2}$.

$$x(t) = \frac{A \sin \delta t}{2\overline{\omega}\delta} \sin \overline{\omega} t.$$

  - ◇ Fast oscillation with frequency $\overline{\omega}$ with amplitude oscillating slowly with frequency δ.
    - ⋆ Beats.

Return

- $\omega = \omega_0$

$$y'' + \omega_0^2 y = A\cos\omega_0 t.$$

  - ◇ An exceptional case. Particular solution

$$x_p(t) = \frac{A}{2\omega_0} t \sin\omega_0 t.$$

  - ◇ Oscillation with increasing amplitude.
  - ◇ First example of *resonance.*
    - ⋆ Driving at the natural frequency can cause oscillations that grow out of control.

Forced, Damped Harmonic Motion

$$x'' + 2cx' + \omega_0^2 x = A\cos\omega t$$

- Homo. equation: $x'' + 2cx' + \omega_0^2 x = 0$
- Ch. polynomial: $P(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2$
- Assume the underdamped case, where $c < \omega_0$.
- Roots $\lambda = -c \pm \sqrt{c^2 - \omega_0^2} = -c \pm i\eta$ where $\eta = \sqrt{\omega_0^2 - c^2}$.
- Fundamental set of solutions $x_1(t) = e^{-ct}\cos\eta t$ and $x_2(t) = e^{-ct}\sin\eta t$

Return

Inhomogeneous equation

$$x'' + 2cx' + \omega_0^2 x = A\cos\omega t$$

- Use the complex method. Solve

$$z'' + 2cz' + \omega_0^2 z = Ae^{i\omega t}.$$

  ◇ Try $z(t) = ae^{i\omega t}$. $x_p = \mathrm{Re}(z)$.

$$\begin{aligned} z'' + 2cz' + \omega_0^2 z &= [(i\omega)^2 + 2c(i\omega) + \omega_0^2]ae^{i\omega t} \\ &= P(i\omega)z \end{aligned}.$$

$$
\begin{aligned}
P(i\omega) &= (i\omega)^2 + 2c(i\omega) + \omega_0^2 \\
&= [\omega_0^2 - \omega^2] + 2ic\omega
\end{aligned}
$$

- Complex solution: $z(t) = \dfrac{1}{P(i\omega)} A e^{i\omega t}$.
- Real solution: $x_p(t) = \mathrm{Re}(z(t))$.

- Example: $x'' + 5x' + 4x = 50\cos 3t$
  - ◇ $P(i\omega) = -5 + 15i$
  - ◇ $z(t) = \dfrac{10}{-1+3i} e^{3it}$

$$= -(1+3i)(\cos 3t + i\sin 3t)$$

$$= -[(\cos 3t - 3\sin 3t) + i(\sin 3t + 3\cos 3t)]$$

  - ◇ $x_p(t) = \mathrm{Re}(z(t))$

$$= 3\sin 3t - \cos 3t.$$

Transfer Function

- Complex solution:

$$z(t) = \frac{1}{P(i\omega)} A e^{i\omega t}$$
$$= H(i\omega) A e^{i\omega t}.$$

- $H(i\omega) = \dfrac{1}{P(i\omega)}$ is called the *transfer function*.
  - ◇ We will write $H(i\omega) = G(\omega)e^{-i\phi(\omega)}$.
    - ⋆ G is the *gain* and ϕ is the *phase*.

- Start with the characteristic polynomial

$$\begin{aligned} P(i\omega) &= (i\omega)^2 + 2c(i\omega) + \omega_0^2 \\ &= [\omega_0^2 - \omega^2] + 2ic\omega \\ &= Re^{i\phi}. \end{aligned}$$

◇ We need $R\cos\phi = \omega_0^2 - \omega^2$ and

$$R\sin\phi = 2c\omega.$$

◇ Thus $R = \sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}$

$$\phi = \operatorname{arccot}\left(\frac{\omega_0^2 - \omega^2}{2c\omega}\right).$$

- Transfer Function

$$H(i\omega) = \frac{1}{P(i\omega)} = \frac{1}{R}e^{-i\phi} = G(\omega)e^{-i\phi}.$$

- The gain $G(\omega) = \dfrac{1}{R} = \dfrac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}}$.
- The phase shift $\phi = \text{arccot}\left(\dfrac{\omega_0^2 - \omega^2}{2c\omega}\right)$.

- The complex particular solution is

$$
\begin{aligned}
z(t) &= H(i\omega)Ae^{i\omega t} \\
&= G(\omega)e^{-i\phi} \cdot Ae^{i\omega t} \\
&= G(\omega)Ae^{i(\omega t-\phi)}.
\end{aligned}
$$

- The real particular solution is

$$
\begin{aligned}
x_p(t) &= \mathrm{Re}(z(t)) \\
&= G(\omega)A\cos(\omega t - \phi).
\end{aligned}
$$

- General Solution

$$
\begin{aligned}
x(t) &= x_p(t) + x_h(t) \\
&= G(\omega) A \cos(\omega t - \phi) \\
&\qquad + e^{-ct}[C_1 \cos \eta t + C_2 \sin \eta t].
\end{aligned}
$$

- *Transient term*.
  - ◇ $x_h(t) = e^{-ct}[C_1 \cos \eta t + C_2 \sin \eta t]$.
- *Steady-state solution.*
  - ◇ $x_p(t) = G(\omega) A \cos(\omega t - \phi)$.

- Example: $x'' + 5x' + 4x = A\cos\omega t$
- $G(\omega) = \dfrac{1}{\sqrt{(4-\omega^2)^2 + 25\omega^2}}$ and

$$\phi = \operatorname{arccot}\left(\frac{4-\omega^2}{5\omega}\right).$$

◇ With $\omega = 3$,

$$G(3) = \frac{1}{5\sqrt{10}} \approx 0.0632$$

$$\phi = \operatorname{arccot}(-3/5) \approx 2.1112.$$

◇ SS solution $x_p(t) = G(3)A\cos(3t - \phi)$.

Steady-State Solution

$$x_p(t) = G(\omega)A\cos(\omega t - \phi).$$

- The forcing function is $A\cos\omega t$.
- The steady-state response is oscillatory.
  - The amplitude is $G(\omega)$ times the amplitude of the forcing term.
  - At the driving frequency.
  - With a phase shift of ϕ/ω.

Gain

$$G(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}}$$

- Set

$$\omega = s\omega_0 \quad \text{or} \quad s = \frac{\omega}{\omega_0}$$

$$c = \frac{D\omega_0}{2} \quad \text{or} \quad D = \frac{2c}{\omega_0}.$$

Then

$$G(\omega) = \frac{1}{\omega_0^2} \frac{1}{\sqrt{(1 - s^2)^2 + D^2 s^2}}$$