

# Math 211

Lecture #36  
Nonlinear Systems

April 18, 2001

## Interacting Species

- Two species with populations  $x_1$  &  $x_2$ .
- Interaction between the species can be helpful or detrimental.
- Model

$$x_1' = r_1 x_1$$

$$x_2' = r_2 x_2$$

- $r_1$  &  $r_2$  are the *reproductive rates*.

Return

## Reproductive Rates

- If  $x_2 = 0$  the reproductive rate for  $x_1$  is

$$r_1 = a_1 - b_1 x_1.$$

- ◊  $a_1 > 0 \Rightarrow$  natural growth.
- ◊  $a_1 < 0 \Rightarrow$  natural decline.
- ◊  $b_1 = 0$  Malthusian growth.
- ◊  $b_1 > 0$  logistic growth.

Return

- If  $x_2 > 0$  the reproductive rate for  $x_1$  is

$$r_1 = a_1 - b_1x_1 + c_1x_2.$$

- ◊  $c_1 > 0 \Rightarrow$  interaction is helpful to  $x_1$ .
- ◊  $c_1 < 0 \Rightarrow$  interaction is detrimental to  $x_1$ .

- The reproduction rate for  $x_2$  is

$$r_2 = a_2 - b_2x_2 + c_2x_1.$$

Return

Interacting species

## Interacting Species Model

$$x_1' = (a_1 - b_1x_1 + c_1x_2)x_1$$

$$x_2' = (a_2 - b_2x_2 + c_2x_1)x_2$$

Return

Logistic/Malthusian

Help/hurt

## Predator Prey Model

Rabbits & foxes, fish & sharks, and cottony cushion scale insect & ladybird beetle.

- $F$  = fish &  $S$  = sharks.

$$F' = (a - bS)F$$

$$S' = (-c + dF)S$$

or

$$F' = (a - eF - bS)F$$

$$S' = (-c + dF)S$$

Return

Interacting species

Logistic/Malthusian

Help/hurt

### Example of Predator Prey

$$F' = (3 - 3S)F$$

$$S' = (-1 + 3F)S$$

or

$$F' = (3 - 3F - 3S)F$$

$$S' = (-1 + 3F)S$$

Return

### Competing Species

Cattle and sheep.

- $x_1$  and  $x_2$  competing for resources.

$$x'_1 = (a_1 - b_1x_1 + c_1x_2)x_1$$

$$x'_2 = (a_2 - b_2x_2 + c_2x_1)x_2$$

- $a_i > 0$ ,  $b_i > 0$ , &  $c_i < 0$

Return

### Example of Competing Species

$$x' = (5 - 2x - y)x$$

$$y' = (7 - 2x - 3y)y$$

Return

Competing species

## Linearization

Principal idea of differential calculus:

- Approximate nonlinear mathematical objects by linear ones.
- Approximate  $f(y) = f(y_0 + h)$  near  $y_0$  by the linear function  $L(h) = f(y_0) + f'(y_0)h$ .

$$f(y_0 + h) = f(y_0) + f'(y_0)h + R(h)$$

$$\text{where } \lim_{h \rightarrow 0} \frac{R(h)}{h} = 0.$$

Return

## Linearization of an ODE

$$y' = f(y)$$

- Assume  $f(y_0) = 0$  and  $f'(y_0) \neq 0$ .
- Set  $y = y_0 + u$ . Get

$$u' = f'(y_0)u + R(u)$$

- Approximate by the linear equation

$$\tilde{u}' = f'(y_0)\tilde{u}$$

Return

Taylor's theorem

- If  $f'(y_0) \neq 0$  the equilibrium point of the linearization at 0 has the same stability properties as that of the nonlinear equation at  $y_0$ .
  - ◊  $f'(y_0) > 0 \Rightarrow y_0$  is unstable.
  - ◊  $f'(y_0) < 0 \Rightarrow y_0$  is asymptotically stable.
- We can solve the linearization explicitly.

Return

## Linearization of a Planar System

$$x' = f(x, y)$$

$$y' = g(x, y)$$

- $(x_0, y_0)$  is an equilibrium point so

$$f(x_0, y_0) = g(x_0, y_0) = 0$$

Return

We have by Taylor's theorem

$$\begin{aligned} f(x_0 + u, y_0 + v) \\ &= \frac{\partial f}{\partial x}(x_0, y_0)u + \frac{\partial f}{\partial y}(x_0, y_0)v + R_f(u, v) \end{aligned}$$

$$\begin{aligned} g(x_0 + u, y_0 + v) \\ &= \frac{\partial g}{\partial x}(x_0, y_0)u + \frac{\partial g}{\partial y}(x_0, y_0)v + R_g(u, v) \end{aligned}$$

$$\text{where } \frac{R_f(u, v)}{\sqrt{u^2 + v^2}} \rightarrow 0 \text{ and } \frac{R_g(u, v)}{\sqrt{u^2 + v^2}} \rightarrow 0$$

Return

System

- Set  $x = x_0 + u$  and  $y = y_0 + v$ . The system becomes

$$u' = \frac{\partial f}{\partial x}(x_0, y_0)u + \frac{\partial f}{\partial y}(x_0, y_0)v + R_f(u, v)$$

$$v' = \frac{\partial g}{\partial x}(x_0, y_0)u + \frac{\partial g}{\partial y}(x_0, y_0)v + R_g(u, v)$$

Return

Taylor's theorem

### Linearization at $(x_0, y_0)$

$$\tilde{u}' = \frac{\partial f}{\partial x}(x_0, y_0)\tilde{u} + \frac{\partial f}{\partial y}(x_0, y_0)\tilde{v}$$

$$\tilde{v}' = \frac{\partial g}{\partial x}(x_0, y_0)\tilde{u} + \frac{\partial g}{\partial y}(x_0, y_0)\tilde{v}$$

- This is a linear system.
  - ◊ We can solve it explicitly.
  - ◊ Does it give information about the original system?

Return      Original system      Nonlinear system      Matrix form

### Matrix Form of the Linearization

Set  $\mathbf{u} = (\tilde{u}, \tilde{v})^T$  and introduce the *Jacobian matrix*

$$J = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}$$

- The linearization becomes

$$\mathbf{u}' = J\mathbf{u}.$$

Return      Linear system      Original system

**Theorem:** Consider the planar system

$$x' = f(x, y)$$

$$y' = g(x, y)$$

where  $f$  and  $g$  are continuously differentiable.

Suppose that  $(x_0, y_0)$  is an equilibrium point. If the linearization at  $(x_0, y_0)$  has a generic equilibrium point at the origin, then the equilibrium point at  $(x_0, y_0)$  is of the same type.

Return      Matrix form      Generic

## Generic Equilibrium Points

- Saddle, nodal source, nodal sink, spiral source, and spiral sink.
  - ◊ All occupy large open subsets of the trace-determinant plane.
- Nongeneric types
  - ◊ Center and eight others. Occupy pieces of the boundaries.

Return

Theorem

## Examples

- Predator prey
- Competing species
- Center

$$x' = y + \alpha x(x^2 + y^2)$$

$$y' = -x + \alpha y(x^2 + y^2)$$

Return

Theorem

Generic