equilibrium solutions are the solutions that can be "seen" in the direction field in Figure 8. They are shown plotted in blue in Figure 10.

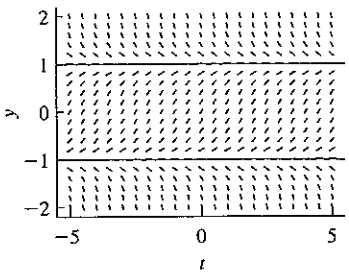

Figure 10. Equilibrium solutions to the equation $y' = 1 - y^2$.

Next we notice that $f(y) = 1 - y^2$ is positive if $-1 < y < 1$ and negative otherwise. Thus, if $y(t)$ is a solution to equation (1.24), and $-1 < y < 1$, then

$$y' = 1 - y^2 > 0.$$

Having a positive derivative, y is an increasing function.

How large can a solution $y(t)$ get? If it gets larger than 1, then $y' = 1 - y^2 < 0$, so $y(t)$ will be decreasing. We cannot complete this line of reasoning at this point, but in Section 2.9 we will develop the argument, and we will be able to conclude that if $y(0) = y_0 > 1$, then $y(t)$ is decreasing and $y(t) \to 1$ as $t \to \infty$.

On the other hand, if $y(0) = y_0$ satisfies $-1 < y_0 < 1$, then $y' = 1 - y^2 > 0$, so $y(t)$ will be increasing. We will again conclude that $y(t)$ increases and approaches 1 as $t \to \infty$. Thus any solution to the equation $y' = 1 - y^2$ with an initial value $y_0 > -1$ approaches 1 as $t \to \infty$.

Finally, if we consider a solution $y(t)$ with $y(0) = y_0 < -1$, then a similar analysis shows that $y'(t) = 1 - y^2 < 0$, so $y(t)$ is decreasing. As $y(t)$ decreases, its derivative $y'(t) = 1 - y^2$ gets more and more negative. Hence, $y(t)$ decreases faster and faster and must approach $-\infty$ as t increases. Typical solutions to equation (1.24) are shown in Figure 11. These solutions were found with a computer, but their qualitative nature can be found simply by looking at the equation.

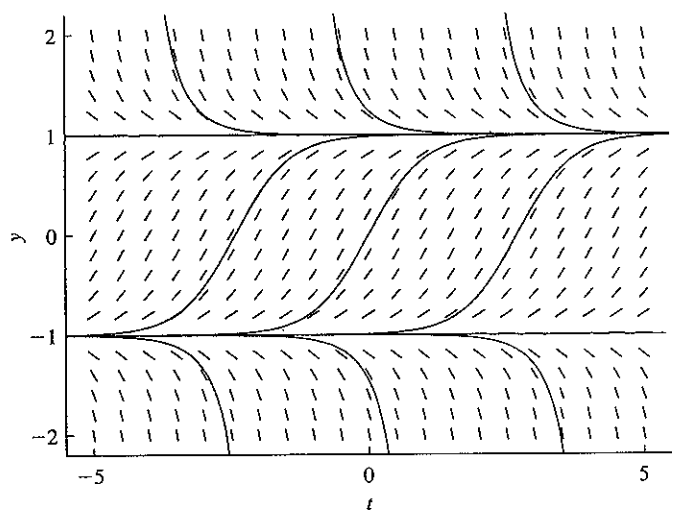

Figure 11. Typical solutions to the equation $y' = 1 - y^2$.

EXERCISES (§2.1)

In Exercises 1 and 2, given the function ϕ, place the ordinary differential equation $\phi(t, y, y') = 0$ in normal form.

1. $\phi(x, y, z) = x^2 z + (1 + x)y$

2. $\phi(x, y, z) = xz - 2y - x^2$

In Exercises 3–6, show that the given solution is a general solution of the differential equation. Use a computer or calculator to sketch the solutions for the given values of the arbitrary constant. Experiment with different intervals for t until you have a plot that shows what you consider to be the most important behavior of the family.

3. $y' = -ty$, $y(t) = Ce^{-(1/2)t^2}$, $C = -3, -2, \dots, 3$

4. $y' + y = 2t$, $y(t) = 2t - 2 + Ce^{-t}$, $C = -3, -2, \dots, 3$

5. $y' + (1/2)y = 2\cos t$, $y(t) = (4/5)\cos t + (8/5)\sin t + Ce^{-(1/2)t}$, $C = -5, -4, \dots, 5$

6. $y' = y(4 - y)$, $y(t) = 4/(1 + Ce^{-4t})$, $C = 1, 2, \dots, 5$

7. A general solution may fail to produce all solutions of a differential equation. In Exercise 6, show that $y = 0$ is a solution of the differential equation, but no value of C in the given general solution will produce this solution.

8. (a) Use implicit differentiation to show that $t^2 + y^2 = C^2$ implicitly defines solutions of the differential equation $t + yy' = 0$.
   (b) Solve $t^2 + y^2 = C^2$ for y in terms of t to provide explicit solutions. Show that these functions are also solutions of $t + yy' = 0$.
   (c) Discuss the interval of existence for each of the solutions in part (b).
   (d) Sketch the solutions in part (b) for $C = 1, 2, 3, 4$.

9. (a) Use implicit differentiation to show that $t^2 - 4y^2 = C^2$ implicitly defines solutions of the differential equation $t - 4yy' = 0$.
   (b) Solve $t^2 - 4y^2 = C^2$ for y in terms of t to provide explicit solutions. Show that these functions are also solutions of $t - 4yy' = 0$.
   (c) Discuss the interval of existence for each of the solutions in part (b).
   (d) Sketch the solutions in part (b) for $C = 1, 2, 3, 4$.

10. Show that $y(t) = 3/(6t - 11)$ is a solution of $y' = -2y^2$, $y(2) = 3$. Sketch this solution and discuss its interval of existence. Include the initial condition on your sketch.

11. Show that $y(t) = 4/(1 - 5e^{-4t})$ is a solution of the initial value problem $y' = y(4 - y)$, $y(0) = -1$. Sketch this solution and discuss its interval of existence. Include the initial condition on your sketch.

In Exercises 12–15, use the given general solution to find a solution of the differential equation having the given initial condition. Sketch the solution, the initial condition, and discuss the solution's interval of existence.

12. $y' + 4y = \cos t$, $y(t) = (4/17)\cos t + (1/17)\sin t + Ce^{-4t}$, $y(0) = -1$

13. $ty' + y = t^2$, $y(t) = (1/3)t^2 + C/t$, $y(1) = 2$

14. $ty' + (t + 1)y = 2te^{-t}$, $y(t) = e^{-t}(t + C/t)$, $y(1) = 1/e$

15. $y' = y(2 + y)$, $y(t) = 2/(-1 + Ce^{-2t})$, $y(0) = -3$

16. Maple, when asked for the solution of the initial value problem $y' = \sqrt{y}$, $y(0) = 1$, returns two solutions: $y(t) = (1/4)(t + 2)^2$ and $y(t) = (1/4)(t - 2)^2$. Present a thorough discussion of this response, including a check and a graph of each solution, interval of existence, and so on. *Hint:* Remember that $\sqrt{a^2} = |a|$.

In Exercises 17–20, plot the direction field for the differential equation by hand. Do this by drawing short lines of the appropriate slope centered at each of the integer valued coordinates (t, y), where $-2 \le t \le 2$ and $-1 \le y \le 1$.

17. $y' = y + t$

18. $y' = y^2 - t$

19. $y' = t\tan(y/2)$

20. $y' = (t^2 y)/(1 + y^2)$

In Exercises 21–24, use a computer to draw a direction field for the given first-order differential equation. Use the indicated bounds for your display window. Obtain a printout and use a pencil to draw a number of possible solution trajectories on the direction field. If possible, check your solutions with a computer.

21. $y' = -ty$, $R = \{(t, y) : -3 \le t \le 3, -5 \le y \le 5\}$

22. $y' = y^2 - t$, $R = \{(t, y) : -2 \le t \le 10, -4 \le y \le 4\}$

23. $y' = t - y + 1$, $R = \{(t, y) : -6 \le t \le 6, -6 \le y \le 6\}$

24. $y' = (y + t)/(y - t)$, $R = \{(t, y) : -5 \le t \le 5, -5 \le y \le 5\}$

For each of the initial value problems in Exercises 25–28 use a numerical solver to plot the solution curve over the indicated interval. Try different display windows by experimenting with the bounds on y. *Note:* Your solver might require that you first place the differential equation in normal form.

25. $y + y' = 2$, $y(0) = 0$, $t \in [-2, 10]$

26. $y' + ty = t^2$, $y(0) = 3$, $t \in [-4, 4]$

27. $y' - 3y = \sin t$, $y(0) = -3$, $t \in [-6\pi, \pi/4]$

28. $y' + (\cos t)y = \sin t$, $y(0) = 0$, $t \in [-10, 10]$

Some solvers allow the user to choose dependent and independent variables. For example, your solver may allow the equation $r' = -2sr + e^{-s}$, but other solvers will insist that you change variables so that the equation reads $y' = -2ty + e^{-t}$, or $y' = -2xy + e^{-x}$, should your solver require t or x as the independent variable. For each of the initial value problems in Exercises 29 and 30, use your solver to plot solution curves over the indicated interval.

29. $r' + xr = \cos(2x)$, $r(0) = -3$, $x \in [-4, 4]$

30. $T' + T = s$, $T(-3) = 0$, $s \in [-5, 5]$

In Exercises 31–34, plot solution curves for each of the initial conditions on one set of axes. Experiment with the different display windows until you find one that exhibits what you feel is all of the important behavior of your solutions. *Note:* Selecting a good display window is an art, a skill developed with experience. Don't become overly frustrated in these first attempts.

31. $y' = y(3 - y)$, $y(0) = -2, -1, 0, 1, 2, 3, 4, 5$

32. $x' - x^2 = t$, $x(0) = -2, 0, 2$, $x(2) = 0$, $x(4) = -3, 0, 3$, $x(6) = 0$

33. $y' = \sin(xy)$, $y(0) = 0.5, 1.0, 1.5, 2.0, 2.5$

34. $x' = -tx$, $x(0) = -3, -2, -1, 0, 1, 2, 3$

35. Bacteria in a petri dish is growing according to the equation

$$\frac{dP}{dt} = 0.44P,$$

where P is the mass of the accumulated bacteria (measured in milligrams) after t days. Suppose that the initial mass of the bacterial sample is 1.5 mg. Use a numerical solver to estimate the amount of bacteria after 10 days.

The integral on the left contains the expression $y'(t)\,dt$. This is inviting us to change the variable of integration to y, since when we do that, we use the equation $dy = y'(t)\,dt$. Making the change of variables leads to

$$\int h(y)\,dy = \int g(t)\,dt. \tag{2.37}$$

Notice the similarity between (2.36) and (2.37). Equation (2.36), which has no meaning by itself, acquires a precise meaning when both sides are integrated. Since this is precisely the next step that we take when solving separable equations, we can be sure that our method is valid.

We mention in closing that the objects in (2.36), $h(y)\,dy$ and $g(t)\,dt$, can be given meaning as formal objects that can be integrated. They are called ***differential forms***, and the special cases like dy and dt are called ***differentials***. The basic formula connecting differentials dy and dt when y is a function of t is

$$dy = y'(t)\,dt,$$

the change-of-variables formula in integration. These techniques will assume greater importance in Section 2.6, where we will deal with exact equations. The use of differential forms is very important in the study of the calculus of functions of several variables and especially in applications to geometry and to parts of physics.

EXERCISES (§2.2)

In Exercises 1–12, find the general solution of the indicated differential equation. If possible, find an explicit solution.

1. $y' = xy$
2. $xy' = 2y$
3. $y' = e^{x-y}$
4. $y' = (1+y^2)e^x$
5. $y' = xy + y$
6. $y' = ye^x - 2e^x + y - 2$
7. $y' = x/(y+2)$
8. $y' = xy/(x-1)$
9. $x^2y' = y\ln y - y'$
10. $xy' - y = 2x^2y$
11. $y^3y' = x + 2y'$
12. $y' = (2xy+2x)/(x^2-1)$

In Exercises 13–18, find the exact solution of the initial value problem. Indicate the interval of existence.

13. $y' = y/x$, $y(1) = -2$
14. $y' = -2t(1+y^2)/y$, $y(0) = 1$
15. $y' = (\sin x)/y$, $y(\pi/2) = 1$
16. $y' = e^{x+y}$, $y(0) = 0$
17. $y' = (1+y^2)$, $y(0) = 1$
18. $y' = x/(1+2y)$, $y(-1) = 0$

In Exercises 19–22, find exact solutions for each given initial condition. State the interval of existence in each case. Plot each exact solution on the interval of existence. Use a numerical solver to duplicate the solution curve for each initial value problem.

19. $y' = x/y$, $y(0) = 1$, $y(0) = -1$
20. $y' = -x/y$, $y(0) = 2$, $y(0) = -2$
21. $y' = 2 - y$, $y(0) = 3$, $y(0) = 1$
22. $y' = (y^2+1)/y$, $y(1) = 2$
23. Suppose that a radioactive substance decays according to the model $N' = N_0e^{-\lambda t}$. Show that the half-life of the radioactive substance is given by the equation

$$T_{1/2} = \frac{\ln 2}{\lambda}. \tag{2.38}$$

24. The half-life of ^{238}U is 4.47×10^7 yr.
    (a) Use equation (2.38) to compute the ***decay constant*** λ for ^{238}U.
    (b) Suppose that 1000 mg of ^{238}U are present initially. Use the equation $N = N_0e^{-\lambda t}$ and the decay constant determined in part (a) to determine the time for this sample to decay to 100 mg.
25. Tritium, ^3H, is an isotope of hydrogen that is sometimes used as a biochemical tracer. Suppose that 100 mg of ^3H decays to 80 mg in 4 hours. Determine the half-life of ^3H.
26. The isotope Technetium 99m is used in medical imaging. It has a half-life of about 6 hours, a useful feature for radioisotopes that are injected into humans. The Technetium, having such a short half-life, is created artificially on scene by harvesting from a more stable isotope, ^{99}Mb. If 10 g of ^{99m}Tc are "harvested" from the Molybdenum, how much of this sample remains after 9 hours?
27. The isotope Iodine 131 is used to destroy tissue in an overactive thyroid gland. It has a half-life of 8.04 days. If a hospital receives a shipment of 500 mg of ^{131}I, how much of the isotope will be left after 20 days?