HW 10:

§ 4.5 33, 37

§ 4.6 11, 13

§ 9.8 39, 43

(§4.5)

Use the technique suggested by Examples 5.23 and 5.26, as well as Exercise 30, to help find particular solutions for the differential equations in Exercises 31–38.

31. $y'' + 2y' + 2y = 2 + \cos 2t$

32. $y'' - y = t - e^{-t}$

33. $y'' + 25y = 2 + 3t + \cos 5t$

34. $y'' + 2y' + y = 3 - e^{-t}$

35. $y'' + 4y' + 3y = \cos 2t + 3\sin 2t$

36. $y'' + 2y' + 2y = 3\cos t - \sin t$

37. $y'' + 4y' + 4y = e^{-2t} + \sin 2t$

38. $y'' + 16y = e^{-4t} + 3\sin 4t$

39. Use the form $y_p(t) = (at + b)e^{-4t}$ in an attempt to find a particular solution of the equation $y'' + 3y' + 2y = te^{-4t}$.

Use an approach similar to that in Exercise 39 to find particular solutions of the equations in Exercises 40–43.

40. $y'' - 3y' + 2y = te^{-3t}$ **41.** $y'' + 2y' + y = t^2e^{-2t}$

42. $y'' + 5y' + 4y = te^{-t}$ **43.** $y'' + 3y' + 2y = t^2e^{-2t}$

44. Use the form $y_p = e^{-2t}(a\cos t + b\sin t)$ in an attempt to find a particular solution of $y'' + 2y' + 2y = e^{-2t}\sin t$.

45. If $z(t) = x(t) + iy(t)$ is a solution of

$$z'' + pz' + qz = Ae^{(a+bi)t},$$

show that $x(t)$ and $y(t)$ are solutions of

$$x'' + px' + qx = Ae^{at}\cos bt$$

and

$$y'' + py' + qy = Ae^{at}\sin bt,$$

respectively.

46. Use the technique suggested by Exercise 45 to find a particular solution of the equation in Exercise 44.

47. Prove that the imaginary part of the solution of $z'' + z' + z = te^{it}$ is a solution of $y'' + y' + y = t\sin t$. Use this idea to find a particular solution of $y'' + y' + y = t\sin t$.

48. Prove Theorem 5.22.

4.6 Variation of Parameters

In this section we introduce a technique called ***variation of parameters***. This technique is used to find a particular solution to more general higher-order equations, provided we know a fundamental set of solutions to the associated homogeneous equation. As we did in the previous section, we will illustrate the method for second-order equations. The method also works for higher-order equations, but it is usually more efficient to solve the associated first-order system using variation of parameters. This will be discussed in a later chapter.

We are interested in solving the equation

$$y'' + p(t)y' + q(t)y = g(t). \tag{6.1}$$

Notice that we are allowing the coefficients $p(t)$ and $q(t)$ to be functions of t. In particular, we are not restricting them to be constants. This might seem to be a great increase in generality, but there is a rather strong constraint. We will have to assume that we have computed a fundamental set of solutions y_1 and y_2 to the associated homogeneous equation

$$y'' + p(t)y' + q(t)y = 0. \tag{6.2}$$

Then the general solution to the homogeneous equation is

$$y_h = C_1y_1 + C_2y_2, \tag{6.3}$$

where C_1 and C_2 are arbitrary constants.

The idea behind variation of parameters is to replace the constants C_1 and C_2 in (6.3) by unknown functions $v_1(t)$ and $v_2(t)$ and look for a particular solution to the inhomogeneous equation (6.1) of the form

$$y_p = v_1y_1 + v_2y_2. \tag{6.4}$$

You will notice the similarity with the method of variation of parameters as it was used to solve first-order linear equations in Chapter 2.

(a) Differentiate y_p:

$$y_p' = (v_1'y_1 + v_2'y_2) + v_1y_1' + v_2y_2'$$

and set the first term on the right equal to zero.

$$v_1'y_1 + v_2'y_2 = 0 \tag{6.17}$$

(b) Take the second derivative of y_p and insert y_p, y_p', and y_p'' into the differential equation. After simplifying, a second equation will appear:

$$v_1'y_1' + v_2'y_2' = g(t). \tag{6.18}$$

(c) Solve (6.17) and (6.18) for v_1' and v_2' by elimination.

(d) Integrate to find v_1 and v_2.

4. Substitute v_1 and v_2 into $y_p = v_1y_1 + v_2y_2$.

It is up to the reader to decide which of the methods to use in step 3.

Although variation of parameters always works theoretically, its success requires a fundamental set of solutions to the homogeneous equation and the ability to compute the integrals in (6.16). The method is of significant theoretical importance, however.

EXERCISES (§4.6)

For Exercises 1–12, find a particular solution to the given second-order differential equation.

1. $y'' + 9y = \tan 3t$
2. $y'' + 4y = \sec 2t$
3. $y'' - y = t + 3$
4. $x'' - 2x' - 3x = 4e^{3t}$
5. $y'' - 2y' + y = e^t$
6. $x'' - 4x' + 4x = e^{2t}$
7. $x'' + x = \tan^2 t$
8. $x'' + x = \sec^2 t$
9. $x'' + x = \sin^2 t$
10. $y'' + 2y' + y = t^5e^{-t}$
11. $y'' + y = \tan t + \sin t + 1$
12. $y'' + y = \sec t + \cos t - 1$
13. Verify that $y_1(t) = t$ and $y_2(t) = t^{-3}$ are solutions to the homogeneous equation

$$t^2y''(t) + 3ty'(t) - 3y(t) = 0.$$

Use variation of parameters to find the general solution to

$$t^2y''(t) + 3ty'(t) - 3y(t) = \frac{1}{t}.$$

14. Verify that $y_1(t) = t^{-1}$ and $y_2(t) = t^{-1}\ln t$ are solutions to the homogeneous equation

$$t^2y''(t) + 3ty'(t) + y(t) = 0.$$

Use variation of parameters to find the general solution to

$$t^2y''(t) + 3ty'(t) + y(t) = \frac{1}{t}.$$

4.7 Forced Harmonic Motion

In this section, we apply the technique of undetermined coefficients to analyze harmonic motion with an external sinusoidal forcing term. We derived the model in Section 4.4, and with a sinusoidal forcing term, equation (4.4) becomes

$$x'' + 2cx' + \omega_0^2x = A\cos\omega t. \tag{7.1}$$

The constant A is the amplitude of the driving force, and ω is the ***driving frequency***. Remember that c is the damping constant and ω_0 is the natural frequency.

To focus our thinking, we may suppose that we have an iron mass m suspended on a spring, with the top of the spring attached to a motor that moves the top of the spring up and down. We can also consider an *RLC* circuit in which the source voltage is sinusoidal.

We will first treat the case with no damping.

are solutions of system (2). Show that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are linearly independent.

(b) Use direct substitution to show that the first component of the general solution $\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t)$ is a solution of $y'' + 4y = 0$.

Use Definition 8.15 and the technique of Example 8.16 to show that each set of functions in Exercises 3–6 is linearly independent.

3. $y_1(t) = e^t$ and $y_2(t) = e^{2t}$
4. $y_1(t) = e^t \cos t$ and $y_2(t) = e^t \sin t$
5. $y_1(t) = \cos t$, $y_2(t) = \sin t$, and $y_3(t) = e^t$
6. $y_1(t) = e^t$, $y_2(t) = te^t$, and $y_3(t) = t^2e^t$

In Exercises 7–12, use Proposition 8.19 and the technique of Example 8.20 to show that the given solutions are linearly independent and form a fundamental set of solutions of the given equation.

7. The equation $y'' + 9y = 0$ has solutions $y_1(t) = \cos 3t$ and $y_2(t) = \sin 3t$.
8. The equation $y'' + 9y' - 10y = 0$ has solutions $y_1(t) = e^{-10t}$ and $y_2(t) = e^t$.
9. The equation $y'' - 4y' + 4y = 0$ has solutions $y_1(t) = e^{2t}$ and $y_2(t) = te^{2t}$.
10. The equation $y''' - 3y'' + 9y' - 27y = 0$ has solutions $y_1(t) = \cos 3t$, $y_2(t) = \sin 3t$, and $y_3(t) = e^{3t}$.
11. The equation $y''' - 3y'' + 3y' - y = 0$ has solutions $y_1(t) = e^t$, $y_2(t) = te^t$, and $y_3(t) = t^2e^t$.
12. The equation $y^{(4)} - 13y'' + 36y = 0$ has solutions $y_1(t) = \cos 3t$, $y_2(t) = \sin 3t$, $y_3(t) = \cos 2t$, and $y_4(t) = \sin 2t$.
13. Consider the equation

$$y''' + ay'' + by' + cy = 0. \qquad (8.38)$$

(a) If $e^{\lambda t}$ is a solution of equation (8.38), provide details showing that $\lambda^3 + a\lambda^2 + b\lambda + c = 0$.

(b) Write the third-order equation (8.38) as a system of first-order equations, placing your answer in the form $\mathbf{x}' = A\mathbf{x}$. Calculate the characteristic polynomial of system $\mathbf{x}' = A\mathbf{x}$ and compare it with the characteristic polynomial of the equation in (8.38).

Each equation in Exercises 14–21 has a characteristic equation possessing distinct real roots. Find the general solution of each equation.

14. $y''' - 2y'' - y' + 2y = 0$
15. $y''' - 3y'' = 4y' - 12y$
16. $y^{(4)} - 5y'' + 4y = 0$
17. $y^{(4)} + 36y = 13y''$
18. $y''' + 2y'' - 5y' - 6y = 0$
19. $y''' + 30y = 4y'' + 11y'$
20. $y^{(5)} + 3y^{(4)} - 5y''' - 15y'' + 4y' + 12y = 0$
21. $y^{(5)} - 4y^{(4)} - 13y''' + 52y'' + 36y' - 144y = 0$

Each equation in Exercises 22–27 has a characteristic equation possessing real roots of various multiplicities. Find the general solution of each equation.

22. $y''' - 3y' + 2y = 0$
23. $y''' + y'' = 8y' + 12y$
24. $y''' + 6y'' + 12y' + 8y = 0$
25. $y''' + 3y'' + 3y' + y = 0$
26. $y^{(5)} + 3y^{(4)} - 6y''' - 10y'' + 21y' - 9y = 0$
27. $y^{(5)} - y^{(4)} - 6y''' + 14y'' - 11y' + 3y = 0$

Each equation in Exercises 28–33 has a characteristic equation possessing some complex zeros, some of which are repeated. Find the general solution of each equation.

28. $y''' - y'' + 4y' - 4y = 0$
29. $y''' + 2y = y''$
30. $y^{(4)} + 17y'' + 16y = 0$
31. $y^{(4)} + y = -2y''$
32. $y^{(5)} - 9y^{(4)} + 34y''' - 66y'' + 65y' - 25y = 0$
33. $y^{(6)} + 3y^{(4)} + 3y'' + y = 0$

Find the solution of each initial-value problem presented in Exercises 34–43.

34. $y'' - 2y' - 3y = 0$, with $y(0) = 4$ and $y'(0) = 0$
35. $y'' + 2y' + 5y = 0$, with $y(0) = 2$ and $y'(0) = 0$
36. $y'' + 4y' + 4y = 0$, with $y(0) = 2$ and $y'(0) = -1$
37. $y'' - 2y' + y = 0$, with $y(0) = 1$ and $y'(0) = 0$
38. $y''' - 4y'' - 7y' + 10y = 0$, with $y(0) = 1$, $y'(0) = 0$, and $y''(0) = -1$
39. $y''' - 7y'' + 11y' - 5y = 0$, with $y(0) = -1$, $y'(0) = 1$, and $y''(0) = 0$
40. $y''' - 2y' + 4y = 0$, $y(0) = 1$, with $y'(0) = -1$, and $y''(0) = 0$
41. $y''' - 6y'' + 12y' - 8y = 0$, with $y(0) = -2$, $y'(0) = 0$, and $y''(0) = 2$
42. $y''' - 3y' + 52y = 0$, $y(0) = 0$, with $y'(0) = -1$, and $y''(0) = 2$
43. $y^{(4)} + 8y'' + 16y = 0$, with $y(0) = 0$, $y'(0) = -1$, $y''(0) = 2$, and $y'''(0) = 0$

In Exercises 44–46, we will verify some of the steps leading to the proof of Theorem 8.33.

44. Verfiy (8.30).
45. Verfiy (8.31). See the definition of a subspace in Definition 5.5 in Section 7.5.
46. Verfiy (8.32). *Hint*: Suppose that there are constants $c_1, \ldots, c_k$ such that $c_1\mathbf{a}_1 + \cdots + c_k\mathbf{a}_k = \mathbf{0}$. Show that $c_1y_1(t) + \cdots + c_ky_k(t) = 0$. Conclude that all of the constants must be equal to zero.