HW 11:

§ 9.6 6, 7, 20, 27, 28

§ 9.8 2

This matrix has a nullspace of dimension 1, so the eigenvalue $\lambda = -1 + i$ has geometric multiplicity 1. Our computer tells us that $\mathbf{w}_1 = (1+i, 0, 2+2i, -1)^T$ is an eigenvector. The corresponding complex-valued solution is

$$\mathbf{z}_1(t) = e^{tA}\mathbf{w}_1 = e^{(-1+i)t}\mathbf{w}_1 = e^{(-1+i)t}\begin{pmatrix} 1+i \\ 0 \\ 2+2i \\ -1 \end{pmatrix}.$$

To find another complex-valued solution, we compute

$$(A - (-1+i)I)^2 = \begin{pmatrix} 2-14i & 3-12i & -2+6i & -4i \\ 8i & -2+6i & -4i & 0 \\ 8-16i & 6-14i & -6+6i & -8i \\ -2-2i & -1 & 1+2i & -2+2i \end{pmatrix}.$$

We look for a vector $\mathbf{w}_2$ in this nullspace that is not an eigenvector and therefore not a multiple of $\mathbf{w}_1$. One choice is $\mathbf{w}_2 = (1+2i, -2-2i, 0, 2)^T$. Since

$$(A - (-1+i)I)^2\mathbf{w}_2 = \mathbf{0},$$

the corresponding solution is

$$\begin{aligned} \mathbf{z}_2(t) = e^{tA}\mathbf{w}_2 &= e^{(-1+i)t}\left(\mathbf{w}_2 + t(A - (-1+i)I)\mathbf{w}_2\right) \\ &= e^{(-1+i)t}\begin{pmatrix} (1+t) + i(2+t) \\ -2-2i \\ 2t + 2it \\ 2-t \end{pmatrix}. \end{aligned}$$

Corresponding to the complex conjugate eigenvalue $-1-i$, we have the conjugate generalized eigenvectors $\overline{\mathbf{w}}_1$ and $\overline{\mathbf{w}}_2$ and the corresponding conjugate solutions $\overline{z}_1$ and $\overline{z}_2$. These are the required four solutions. To find real solutions, we use the real and imaginary parts of the complex solutions. If $\mathbf{z}_1 = \mathbf{x}_1 + i\mathbf{y}_1$ and $\mathbf{z}_2 = \mathbf{x}_2 + i\mathbf{y}_2$, then $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{y}_1$, and $\mathbf{y}_2$ are the four needed real solutions. ●

EXERCISES (§9.6

Use Definition 6.5 to calculate e^A for the matrices in Exercises 1–4.

1. $A = \begin{pmatrix} -2 & -4 \\ 1 & 2 \end{pmatrix}$ **2.** $A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$

3. $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ **4.** $A = \begin{pmatrix} -2 & 1 & -3 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

5. Suppose that the matrix A satisfies $A^2 = \alpha A$, where $\alpha \neq 0$.

(a) Use Definition 6.5 to show that

$$e^{tA} = I + \frac{e^{\alpha t} - 1}{\alpha}A.$$

(b) Use part (a) to compute e^{tA} for

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

6. There are many important series in mathematics, such as the exponential series. For example,

$$\cos t = \sum_{k=0}^{\infty}(-1)^k\frac{t^{2k}}{(2k)!} = 1 - \frac{t^2}{2!} - \frac{t^4}{4!} + \cdots \quad \text{and}$$

$$\sin t = \sum_{k=0}^{\infty}(-1)^k\frac{t^{2k+1}}{(2k+1)!} = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots.$$

Use these infinite series together with Definition 6.5 to show that

$$e^{t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

7. Use the re

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Hint: $A =$

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11. $A = \Big($ -2 / 0

13. Let A be a

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In Exercises 14

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of Exercise 13

14. $A = \Big($ -2 / -1

16. $A = \Big($ 0 / -1

Each of the ma

value λ. In eac

$(A - \lambda I)^k = 0$

$e^{tA} = e^{\lambda t}$

to compute e^{tA}.

18. $A = \Big($ -1 / -1 / -2

7. Use the result of Exercise 6 to show that if

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

then

$$e^{tA} = e^{at}\begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix}.$$

Hint: $A = aI + b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

8. If

$$A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix},$$

find e^{tA}. *Hint*: See the hint for Exercise 7.

9. Let

$$A = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}.$$

(a) Show that $AB \neq BA$.

(b) Evaluate e^{A+B}. *Hint*: This is a simple computation if you use Exercise 7.

(c) Use Definition 6.5 to evaluate e^A and e^B. Use these results to compute $e^A e^B$ and compare this with the result found in part (b). What have you learned from this exercise?

10. If $A = PDP^{-1}$, prove that $e^{tA} = Pe^{tD}P^{-1}$.

Use the results of Exercise 53 of Section 9.1 and Exercise 10 to calculate e^{tA} for each matrix in Exercises 11–12.

11. $A = \begin{pmatrix} -2 & 6 \\ 0 & -1 \end{pmatrix}$ 12. $A = \begin{pmatrix} -2 & 0 \\ -3 & -3 \end{pmatrix}$

13. Let A be a 2×2 matrix with a single eigenvalue λ of algebraic multiplicity 2 and geometric multiplicity 1. Prove that

$$e^{At} = e^{\lambda t}\left[I + (A - \lambda I)t\right].$$

In Exercises 14–17, each matrix has an eigenvalue of algebraic multiplicity 2 but geometric multiplicity 1. Use the technique of Exercise 13 to compute e^{tA}.

14. $A = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}$ 15. $A = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$

16. $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$ 17. $A = \begin{pmatrix} -3 & -1 \\ 4 & 1 \end{pmatrix}$

Each of the matrices in Exercises 18–25 has only one eigenvalue λ. In each exercise, determine the smallest k such that $(A - \lambda I)^k = 0$. The use the fact that

$$e^{tA} = e^{\lambda t}\left[I + t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \cdots\right]$$

to compute e^{tA}.

18. $A = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & -1 \\ -2 & 4 & -3 \end{pmatrix}$ 19. $A = \begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 2 & -2 \end{pmatrix}$

20. $A = \begin{pmatrix} -2 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & -4 \end{pmatrix}$ 21. $A = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 1 & -2 \end{pmatrix}$

22. $A = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{pmatrix}$

23. $A = \begin{pmatrix} -5 & 0 & -1 & 4 \\ -4 & 0 & 1 & 5 \\ 4 & -4 & -5 & -4 \\ 0 & -1 & -1 & -2 \end{pmatrix}$

24. $A = \begin{pmatrix} 0 & 4 & 5 & -2 \\ 1 & -5 & -7 & 3 \\ 0 & 2 & 3 & -1 \\ 3 & -10 & -13 & 6 \end{pmatrix}$

25. $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -9 & 4 & 1 & 4 \\ 13 & -3 & -1 & -5 \\ 2 & -1 & 0 & 0 \end{pmatrix}$

Do the following for each of the matrices in Exercises 26–33. Exercises 26–29 can be done by hand, but you should use a computer for the rest.

(i) Find the eigenvalues.

(ii) For each eigenvalue, find the algebraic and the geometric multiplicities.

(iii) For each eigenvalue λ, find the smallest integer k such that the dimension of the nullspace of $(A - \lambda I)^k$ is equal to the algebraic multiplicity.

(iv) For each eigenvalue λ, find q linearly independent generalized eigenvectors, where q is the algebraic multiplicity of λ.

(v) Verify that the collection of the generalized eigenvectors you find in part (iv) for all of the eigenvalues is linearly independent.

(vi) Find a fundamental set of solutions for the system $\mathbf{y}' = A\mathbf{y}$.

26. $A = \begin{pmatrix} -2 & 1 & -1 \\ 1 & -3 & 0 \\ 3 & -5 & 0 \end{pmatrix}$ 27. $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & -2 \\ 0 & 0 & 2 \end{pmatrix}$

28. $A = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 0 & 1 \end{pmatrix}$ 29. $A = \begin{pmatrix} -1 & 0 & 0 \\ 2 & -5 & -1 \\ 0 & 4 & -1 \end{pmatrix}$

30. $A = \begin{pmatrix} 11 & -42 & 4 & 28 \\ -12 & 39 & -4 & -28 \\ 0 & 0 & -1 & 0 \\ -24 & 81 & -8 & -57 \end{pmatrix}$

31. $A = \begin{pmatrix} 18 & -7 & 24 & 24 \\ 15 & -8 & 20 & 16 \\ 0 & 0 & -1 & 0 \\ -12 & 4 & -15 & -17 \end{pmatrix}$

Now suppose that $\lambda = \alpha + i\beta$ is a complex root of multiplicity q. Then $\overline{\lambda} = \alpha - i\beta$ is also a complex root of multiplicity q. By the same reasoning we used in the real case, we get complex conjugate pairs of solutions

$$z(t) = t^k e^{\lambda t} \quad \text{and} \quad \overline{z}(t) = t^k e^{\overline{\lambda} t}, \quad \text{where} \quad k = 0, 1, \dots, q-1.$$

To find the real solutions, we take the real and imaginary parts,

$$x(t) = t^k e^{\alpha t} \cos \beta t \quad \text{and} \quad y(t) = t^k e^{\alpha t} \sin \beta t, \quad \text{where} \quad k = 0, 1, \dots, q-1.$$

Let's summarize the facts about the solutions associated with the complex roots.

THEOREM 8.36 If $\lambda = \alpha + i\beta$ is a complex root of the characteristic polynomial with multiplicity q, then so is $\overline{\lambda} = \alpha - i\beta$. In addition,

$$x_1(t) = e^{\alpha t} \cos \beta t,\ x_2(t) = t e^{\alpha t} \cos \beta t,\ \dots,\ \text{and } x_q(t) = t^{q-1} e^{\alpha t} \cos \beta t$$
$$y_1(t) = e^{\alpha t} \sin \beta t,\ y_2(t) = t e^{\alpha t} \sin \beta t,\ \dots,\ \text{and } y_q(t) = t^{q-1} e^{\alpha t} \sin \beta t$$

are $2q$ linearly independent solutions. ■

Example 8.37 Find a fundamental set of solutions to

$$y'''' + 4y''' + 14y'' + 20y' + 25y = 0.$$

The characteristic polynomial is

$$\lambda^4 + 4\lambda^3 + 14\lambda^2 + 20\lambda + 25 = (\lambda^2 + 2\lambda + 5)^2.$$

Consequently, we have roots $-1 \pm 2i$, each of multiplicity 2. Thus, we have solutions

$$y_1(t) = e^{-t} \cos 2t, \quad y_2(t) = e^{-t} \sin 2t, \quad y_3(t) = t e^{-t} \cos 2t, \quad \text{and} \quad y_4(t) = t e^{-t} \sin 2t.$$ ●

EXERCISES (§9.8)

1. The function $y(t)$ is a solution of the homogeneous equation $y'' - 2y' - 3y = 0$ if and only if

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

is a solution of

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix} \mathbf{x}.$$

(a) Use direct substitution to show that

$$\mathbf{x}_1(t) = \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$$

are solutions of system (1). Show that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are linearly independent.

(b) Use direct substitution to show that the first component of the general solution $\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t)$ is a solution of $y'' - 2y' - 3y = 0$.

2. The function $y(t)$ is a solution of the homogeneous equation $y'' + 4y = 0$ if and only if

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

is a solution of

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \mathbf{x}.$$

(a) Use direct substitution to show that

$$\mathbf{x}_1(t) = \begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix}$$

are solutions of system (2). Show that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are linearly independent.

(b) Use direct substitution to show that the first component of the general solution $\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t)$ is a solution of $y'' + 4y = 0$.

Use Definition 8.15 and the technique of Example 8.16 to show that each set of functions in Exercises 3–6 is linearly independent.

3. $y_1(t) = e^t$ and $y_2(t) = e^{2t}$
4. $y_1(t) = e^t \cos t$ and $y_2(t) = e^t \sin t$
5. $y_1(t) = \cos t$, $y_2(t) = \sin t$, and $y_3(t) = e^t$
6. $y_1(t) = e^t$, $y_2(t) = te^t$, and $y_3(t) = t^2e^t$

In Exercises 7–12, use Proposition 8.19 and the technique of Example 8.20 to show that the given solutions are linearly independent and form a fundamental set of solutions of the given equation.

7. The equation $y'' + 9y = 0$ has solutions $y_1(t) = \cos 3t$ and $y_2(t) = \sin 3t$.
8. The equation $y'' + 9y' - 10y = 0$ has solutions $y_1(t) = e^{-10t}$ and $y_2(t) = e^t$.
9. The equation $y'' - 4y' + 4y = 0$ has solutions $y_1(t) = e^{2t}$ and $y_2(t) = te^{2t}$.
10. The equation $y''' - 3y'' + 9y' - 27y = 0$ has solutions $y_1(t) = \cos 3t$, $y_2(t) = \sin 3t$, and $y_3(t) = e^{3t}$.
11. The equation $y''' - 3y'' + 3y' - y = 0$ has solutions $y_1(t) = e^t$, $y_2(t) = te^t$, and $y_3(t) = t^2e^t$.
12. The equation $y^{(4)} - 13y'' + 36y = 0$ has solutions $y_1(t) = \cos 3t$, $y_2(t) = \sin 3t$, $y_3(t) = \cos 2t$, and $y_4(t) = \sin 2t$.
13. Consider the equation

$$y''' + ay'' + by' + cy = 0. \tag{8.38}$$

(a) If $e^{\lambda t}$ is a solution of equation (8.38), provide details showing that $\lambda^3 + a\lambda^2 + b\lambda + c = 0$.

(b) Write the third-order equation (8.38) as a system of first-order equations, placing your answer in the form $\mathbf{x}' = A\mathbf{x}$. Calculate the characteristic polynomial of system $\mathbf{x}' = A\mathbf{x}$ and compare it with the characteristic polynomial of the equation in (8.38).

Each equation in Exercises 14–21 has a characteristic equation possessing distinct real roots. Find the general solution of each equation.

14. $y''' - 2y'' - y' + 2y = 0$
15. $y''' - 3y'' = 4y' - 12y$
16. $y^{(4)} - 5y'' + 4y = 0$
17. $y^{(4)} + 36y = 13y''$
18. $y''' + 2y'' - 5y' - 6y = 0$
19. $y''' + 30y = 4y'' + 11y'$
20. $y^{(5)} + 3y^{(4)} - 5y''' - 15y'' + 4y' + 12y = 0$
21. $y^{(5)} - 4y^{(4)} - 13y''' + 52y'' + 36y' - 144y = 0$

Each equation in Exercises 22–27 has a characteristic equation possessing real roots of various multiplicities. Find the general solution of each equation.

22. $y''' - 3y' + 2y = 0$
23. $y''' + y'' = 8y' + 12y$
24. $y''' + 6y'' + 12y' + 8y = 0$
25. $y''' + 3y'' + 3y' + y = 0$
26. $y^{(5)} + 3y^{(4)} - 6y''' - 10y'' + 21y' - 9y = 0$
27. $y^{(5)} - y^{(4)} - 6y''' + 14y'' - 11y' + 3y = 0$

Each equation in Exercises 28–33 has a characteristic equation possessing some complex zeros, some of which are repeated. Find the general solution of each equation.

28. $y''' - y'' + 4y' - 4y = 0$
29. $y''' + 2y = y''$
30. $y^{(4)} + 17y'' + 16y = 0$
31. $y^{(4)} + y = -2y''$
32. $y^{(5)} - 9y^{(4)} + 34y''' - 66y'' + 65y' - 25y = 0$
33. $y^{(6)} + 3y^{(4)} + 3y'' + y = 0$

Find the solution of each initial-value problem presented in Exercises 34–43.

34. $y'' - 2y' - 3y = 0$, with $y(0) = 4$ and $y'(0) = 0$
35. $y'' + 2y' + 5y = 0$, with $y(0) = 2$ and $y'(0) = 0$
36. $y'' + 4y' + 4y = 0$, with $y(0) = 2$ and $y'(0) = -1$
37. $y'' - 2y' + y = 0$, with $y(0) = 1$ and $y'(0) = 0$
38. $y''' - 4y'' - 7y' + 10y = 0$, with $y(0) = 1$, $y'(0) = 0$, and $y''(0) = -1$
39. $y''' - 7y'' + 11y' - 5y = 0$, with $y(0) = -1$, $y'(0) = 1$, and $y''(0) = 0$
40. $y''' - 2y' + 4y = 0$, $y(0) = 1$, with $y'(0) = -1$, and $y''(0) = 0$
41. $y''' - 6y'' + 12y' - 8y = 0$, with $y(0) = -2$, $y'(0) = 0$, and $y''(0) = 2$
42. $y''' - 3y' + 52y = 0$, $y(0) = 0$, with $y'(0) = -1$, and $y''(0) = 2$
43. $y^{(4)} + 8y'' + 16y = 0$, with $y(0) = 0$, $y'(0) = -1$, $y''(0) = 2$, and $y'''(0) = 0$

In Exercises 44–46, we will verify some of the steps leading to the proof of Theorem 8.33.

44. Verfiy (8.30).
45. Verfiy (8.31). See the definition of a subspace in Definition 5.5 in Section 7.5.
46. Verfiy (8.32). *Hint*: Suppose that there are constants $c_1, \ldots, c_k$ such that $c_1\mathbf{a}_1 + \cdots + c_k\mathbf{a}_k = \mathbf{0}$. Show that $c_1y_1(t) + \cdots + c_ky_k(t) = 0$. Conclude that all of the constants must be equal to zero.