HW 2:

§2.4. 5, 15, 19

§2.6. 16, 28

Extra: (1) Draw Direction Field of

$$x' = \frac{2tx}{1+x},$$

using Human Method.

(2) And sketch the integral curve of this direction field passing through $x(0)=1$.

(3) Solve the initial value problem

$$x' = \frac{2tx}{1+x}, \quad x(0)=1.$$

Draw the curve $x=x(t)$ on the tx-plane.

(Rmk: You can compare your answer in (3) with in (2).)

EXERCISES

In Exercises 1–12, find the general solution of the first-order, linear equation.

1. $y' + y = 2$
2. $y' - 3y = 5$
3. $y' + (2/x)y = (\cos x)/x^2$
4. $y' + 2ty = 5t$
5. $x' - 2x/(t+1) = (t+1)^2$
6. $tx' = 4x + t^4$
7. $(1+x)y' + y = \cos x$
8. $(1+x^3)y' = 3x^2y + x^2 + x^5$
9. $L(di/dt) + Ri = E$, L, R, E real constants
10. $y' = my + c_1e^{mx}$, m, c_1 real constants
11. $y' = \cos x - y \sec x$
12. $x' - (n/t)x = e^t t^n$, n an positive integer
13. (a) The differential equation $y' + y\cos x = \cos x$ is linear. Use the integrating factor technique of this section to find the general solution.

    (b) The equation $y' + y\cos x = \cos x$ is also separable. Use the separation of variables technique to solve the equation and discuss any discrepancies (if any) between this solution and the solution found in part (a).

In Exercises 14–17, find the solution of the initial value problem.

14. $y' = y + 2xe^{2x}$, $y(0) = 3$
15. $(x^2+1)y' + 3xy = 6x$, $y(0) = -1$
16. $(1+t^2)y' + 4ty = (1+t^2)^{-2}$, $y(1) = 0$
17. $x' + x\cos t = \frac{1}{2}\sin 2t$, $x(0) = 1$

In Exercises 18–21, find the solution of the initial value problem. Discuss the interval of existence and provide a sketch of your solution.

18. $xy' + 2y = \sin x$, $y(\pi/2) = 0$
19. $(2x+3)y' = y + (2x+3)^{1/2}$, $y(-1) = 0$
20. $y' = \cos x - y\sec x$, $y(0) = 1$
21. $(1+t)x' + x = \cos t$, $x(-\pi/2) = 0$
22. The presence of nonlinear terms prevents us from using the technique of this section. In special cases, a change of variable will transform the nonlinear equation into one that is linear. The equation known as ***Bernoulli's equation***,

    $$x' = a(t)x + f(t)x^n, \quad n \neq 0, 1,$$

    was proposed for solution by James Bernoulli in December 1695. In 1696, Leibniz pointed out that the equation can be reduced to a linear equation by taking x^{1-n} as the dependent variable. Show that the change of variable, $z = x^{1-n}$, will transform the nonlinear Bernoulli equation into the linear equation

    $$z' = (1-n)a(t)z + (1-n)f(t).$$

    Hint: If $z = x^{1-n}$, then $dz/dt = (dz/dx)(dx/dt) = (1-n)x^{-n}(dx/dt)$.

In Exercises 23–26, use the technique of Exercise 22 to transform the Bernoulli equation into a linear equation. Find the general solution of the resulting linear equation.

23. $y' + x^{-1}y = xy^2$
24. $y' + y = y^2$
25. $xy' + y = x^4y^3$
26. $P' = aP - bP^2$
27. The equation

    $$\frac{dy}{dt} + \psi y^2 + \phi y + \chi = 0,$$

    where ψ, ϕ, and χ are functions of t, is called the ***generalized Riccati equation***. In general, the equation is not integrable by quadratures. However, suppose that one solution, say $y = y_1$, is known.

    (a) Show that the substitution $y = y_1 + z$ reduces the generalized Riccati equation to

    $$\frac{dz}{dt} + (2y_1\psi + \phi)z + \psi z^2 = 0,$$

    which is an instance of Bernoulli's equation (see Exercise 22).

    (b) Use the fact that $y_1 = 1/t$ is a particular solution of

    $$\frac{dy}{dt} = -\frac{1}{t^2} - \frac{y}{t} + y^2$$

    to find the equation's general solution.
28. Suppose that you have a closed system containing 1000 individuals. A flu epidemic starts. Let $N(t)$ represent the number of infected individuals in the closed system at time t. Assume that the rate at which the number of infected individuals is changing is jointly proportional to the number of infected individuals and to the number of uninfected individuals. Furthermore, suppose that when 100 individuals are infected, the rate at which individuals are becoming infected is 90 individuals per day. If 20 individuals are infected at time $t = 0$, when will 90% of the population be infected? *Hint:* The assumption here is that there are only healthy individuals and sick individuals. Furthermore, the resulting model can be solved using the technique introduced in Exercise 22.
29. In Exercise 33 of Section 2.2, the time of death of a murder victim is determined using Newton's law of cooling. In particular it was discovered that the proportionality constant in Newton's law was $k = \ln(5/4) \approx 0.223$. Suppose we discover another murder victim at midnight with a body temperature of 31°C. However, this time the air temperature at midnight is 0°C, and is falling at a constant rate of 1°C per hour. At what time did the victim die? (Remember that the normal body temperature is 37°C.)

In Exercises 30–35, use the variation of parameters technique to find the general solution of the given differential equation.

EXERCISES

In Exercises 1–8, calculate the differential dF for the given function F.

1. $F(x, y) = 2xy + y^2$
2. $F(x, y) = x^2 - xy + y^2$
3. $F(x, y) = \sqrt{x^2 + y^2}$
4. $F(x, y) = 1/\sqrt{x^2 + y^2}$
5. $F(x, y) = xy + \tan^{-1}(y/x)$
6. $F(x, y) = \ln(xy) + x^2y^3$
7. $F(x, y) = \ln(x^2 + y^2) + x/y$
8. $F(x, y) = \tan^{-1}(x/y) + y^4$

In Exercises 9–21, determine which of the equations are exact and solve the ones that are.

9. $(2x + y)\,dx + (x - 6y)\,dy = 0$
10. $(1 - y\sin x)\,dx + (\cos x)\,dy = 0$
11. $\left(1 + \dfrac{y}{x}\right) dx - \dfrac{1}{x}\,dy = 0$
12. $\dfrac{x}{\sqrt{x^2 + y^2}}\,dx + \dfrac{y}{\sqrt{x^2 + y^2}}\,dy = 0$
13. $\dfrac{dy}{dx} = \dfrac{3x^2 + y}{3y^2 - x}$
14. $\dfrac{dy}{dx} = \dfrac{x}{x - y}$
15. $(u + v)\,du + (u - v)\,dv = 0$
16. $\dfrac{2u}{u^2 + v^2}\,du + \dfrac{2v}{u^2 + v^2}\,dv = 0$
17. $\dfrac{dr}{ds} = \dfrac{\ln s}{r/s - 2s}$
18. $\dfrac{dy}{du} = \dfrac{2 - y/u}{\ln u}$
19. $\sin 2t\,dx + (2x\cos 2t - 2t)\,dt = 0$
20. $2xy^2 + 4x^3 + 2x^2y\,\dfrac{dy}{dx} = 0$
21. $(2r + \ln y)\,dr + ry\,dy = 0$

In Exercises 22–25, the equations are not exact. However, if you multiply by the given integrating factor, then you can solve the resulting exact equation.

22. $(y^2 - xy)\,dx + x^2\,dy = 0, \quad \mu(x, y) = \dfrac{1}{xy^2}$
23. $(x^2y^2 - 1)y\,dx + (1 + x^2y^2)x\,dy = 0, \quad \mu(x, y) = \dfrac{1}{xy}$
24. $3(y + 1)\,dx - 2x\,dy = 0, \quad \mu(x, y) = \dfrac{y + 1}{x^4}$
25. $(x^2 + y^2 - x)\,dx - y\,dy = 0, \quad \mu(x, y) = \dfrac{1}{x^2 + y^2}$
26. Suppose that $y\,dx + (x^2y - x)\,dy = 0$ has an integrating factor that is a function of x alone [i.e., $\mu = \mu(x)$]. Find the integrating factor and use it to solve the differential equation.
27. Suppose that $(xy - 1)\,dx + (x^2 - xy)\,dy = 0$ has an integrating factor that is a function of x alone [i.e., $\mu = \mu(x)$]. Find the integrating factor and use it to solve the differential equation.
28. Suppose that $2y\,dx + (x + y)\,dy = 0$ has an integrating factor that is a function of y alone [i.e., $\mu = \mu(y)$]. Find the integrating factor and use it to solve the differential equation.
29. Suppose that $(y^2 + 2xy)\,dx - x^2\,dy = 0$ has an integrating factor that is a function of y alone [i.e., $\mu = \mu(y)$]. Find the integrating factor and use it to solve the differential equation.
30. Consider the differential equation $2y\,dx + 3x\,dx = 0$. Determine conditions on a and b so that $\mu(x, y) = x^a y^b$ is an integrating factor. Find a particular integrating factor and use it to solve the differential equation.

The equations in Exercises 31–34 each have the form $P(x, y)\,dx + Q(x, y)\,dy = 0$. In each case, show that P and Q are homogeneous of the same degree. State that degree.

31. $(x + y)\,dx + (x - y)\,dy = 0$
32. $(x^2 - xy - y^2)\,dx + 4xy\,dy = 0$
33. $\left(x - \sqrt{x^2 + y^2}\right) dx - y\,dy = 0$
34. $(\ln x - \ln y)\,dx + dy = 0$

Find the general solution of each homogeneous equation in Exercises 35–39.

35. $(x^2 + y^2)\,dx - 2xy\,dy = 0$
36. $(x + y)\,dx + (y - x)\,dy = 0$
37. $(3x + y)\,dx + x\,dy = 0$
38. $\dfrac{dy}{dx} = \dfrac{y(x^2 + y^2)}{xy^2 - 2x^3}$
39. $x^2y' = 2y^2 - x^2$
40. $(y + 2xe^{-y/x})\,dx - x\,dy = 0$
41. In Figure 8, a goose starts in flight a miles due east of its nest. Assume that the goose maintains constant flight speed (relative to the air) so that it is always flying directly toward its nest. The wind is blowing due north at w miles per hour. Figure 8 shows a coordinate frame with the nest at $(0, 0)$ and the goose at (x, y). It is easily seen (but you should verify it yourself) that

$$\frac{dx}{dt} = -v_0\cos\theta,$$
$$\frac{dy}{dt} = w - v_0\sin\theta.$$

(a) Show that

$$\frac{dy}{dx} = \frac{y - k\sqrt{x^2 + y^2}}{x}, \tag{6.43}$$

where $k = w/v_0$, the ratio of the wind speed to the speed of the goose.