HW 3:

§2.6. 33, 38

§2.7. 12, 23, 26, 32

EXERCISES

In Exercises 1–8, calculate the differential dF for the given function F.

1. $F(x, y) = 2xy + y^2$
2. $F(x, y) = x^2 - xy + y^2$
3. $F(x, y) = \sqrt{x^2 + y^2}$
4. $F(x, y) = 1/\sqrt{x^2 + y^2}$
5. $F(x, y) = xy + \tan^{-1}(y/x)$
6. $F(x, y) = \ln(xy) + x^2y^3$
7. $F(x, y) = \ln(x^2 + y^2) + x/y$
8. $F(x, y) = \tan^{-1}(x/y) + y^4$

In Exercises 9–21, determine which of the equations are exact and solve the ones that are.

9. $(2x + y)\,dx + (x - 6y)\,dy = 0$
10. $(1 - y\sin x)\,dx + (\cos x)\,dy = 0$
11. $\left(1 + \frac{y}{x}\right) dx - \frac{1}{x}\,dy = 0$
12. $\frac{x}{\sqrt{x^2 + y^2}}\,dx + \frac{y}{\sqrt{x^2 + y^2}}\,dy = 0$
13. $\frac{dy}{dx} = \frac{3x^2 + y}{3y^2 - x}$
14. $\frac{dy}{dx} = \frac{x}{x - y}$
15. $(u + v)\,du + (u - v)\,dv = 0$
16. $\frac{2u}{u^2 + v^2}\,du + \frac{2v}{u^2 + v^2}\,dv = 0$
17. $\frac{dr}{ds} = \frac{\ln s}{r/s - 2s}$
18. $\frac{dy}{du} = \frac{2 - y/u}{\ln u}$
19. $\sin 2t\,dx + (2x\cos 2t - 2t)\,dt = 0$
20. $2xy^2 + 4x^3 + 2x^2y\,\frac{dy}{dx} = 0$
21. $(2r + \ln y)\,dr + ry\,dy = 0$

In Exercises 22–25, the equations are not exact. However, if you multiply by the given integrating factor, then you can solve the resulting exact equation.

22. $(y^2 - xy)\,dx + x^2\,dy = 0, \quad \mu(x, y) = \frac{1}{xy^2}$
23. $(x^2y^2 - 1)y\,dx + (1 + x^2y^2)x\,dy = 0, \quad \mu(x, y) = \frac{1}{xy}$
24. $3(y + 1)\,dx - 2x\,dy = 0, \quad \mu(x, y) = \frac{y + 1}{x^4}$
25. $(x^2 + y^2 - x)\,dx - y\,dy = 0, \quad \mu(x, y) = \frac{1}{x^2 + y^2}$
26. Suppose that $y\,dx + (x^2y - x)\,dy = 0$ has an integrating factor that is a function of x alone [i.e., $\mu = \mu(x)$]. Find the integrating factor and use it to solve the differential equation.
27. Suppose that $(xy - 1)\,dx + (x^2 - xy)\,dy = 0$ has an integrating factor that is a function of x alone [i.e., $\mu = \mu(x)$]. Find the integrating factor and use it to solve the differential equation.
28. Suppose that $2y\,dx + (x + y)\,dy = 0$ has an integrating factor that is a function of y alone [i.e., $\mu = \mu(y)$]. Find the integrating factor and use it to solve the differential equation.
29. Suppose that $(y^2 + 2xy)\,dx - x^2\,dy = 0$ has an integrating factor that is a function of y alone [i.e., $\mu = \mu(y)$]. Find the integrating factor and use it to solve the differential equation.
30. Consider the differential equation $2y\,dx + 3x\,dx = 0$. Determine conditions on a and b so that $\mu(x, y) = x^a y^b$ is an integrating factor. Find a particular integrating factor and use it to solve the differential equation.

The equations in Exercises 31–34 each have the form $P(x, y)\,dx + Q(x, y)\,dy = 0$. In each case, show that P and Q are homogeneous of the same degree. State that degree.

31. $(x + y)\,dx + (x - y)\,dy = 0$
32. $(x^2 - xy - y^2)\,dx + 4xy\,dy = 0$
33. $\left(x - \sqrt{x^2 + y^2}\right) dx - y\,dy = 0$
34. $(\ln x - \ln y)\,dx + dy = 0$

Find the general solution of each homogeneous equation in Exercises 35–39.

35. $(x^2 + y^2)\,dx - 2xy\,dy = 0$
36. $(x + y)\,dx + (y - x)\,dy = 0$
37. $(3x + y)\,dx + x\,dy = 0$
38. $\frac{dy}{dx} = \frac{y(x^2 + y^2)}{xy^2 - 2x^3}$
39. $x^2y' = 2y^2 - x^2$
40. $(y + 2xe^{-y/x})\,dx - x\,dy = 0$
41. In Figure 8, a goose starts in flight a miles due east of its nest. Assume that the goose maintains constant flight speed (relative to the air) so that it is always flying directly toward its nest. The wind is blowing due north at w miles per hour. Figure 8 shows a coordinate frame with the nest at (0, 0) and the goose at (x, y). It is easily seen (but you should verify it yourself) that

$$\frac{dx}{dt} = -v_0\cos\theta,$$
$$\frac{dy}{dt} = w - v_0\sin\theta.$$

(a) Show that

$$\frac{dy}{dx} = \frac{y - k\sqrt{x^2 + y^2}}{x}, \qquad (6.43)$$

where $k = w/v_0$, the ratio of the wind speed to the speed of the goose.

near the indicated point in the graph on the left. Notice that with this magnification, the solution curves are distinct, as uniqueness requires.

EXERCISES

Which of the initial value problems in Exercises 1–6 are guaranteed a unique solution by the hypotheses of Theorem 7.16? Justify your answer.

1. $y' = 4 + y^2, \quad y(0) = 1$
2. $y' = \sqrt{y}, \quad y(4) = 0$
3. $y' = t \tan^{-1} y, \quad y(0) = 2$
4. $\omega' = \omega \sin \omega + s, \quad \omega(0) = -1$
5. $x' = \dfrac{t}{x+1}, \quad x(0) = 0$
6. $y' = \dfrac{1}{x} y + 2, \quad y(0) = 1$

For each differential equation in Exercises 7–8, perform each of the following tasks.

(i) Find the general solution of the differential equation. Sketch several members of the family of solutions portrayed by the general solution.

(ii) Show that there is no solution satisfying the given initial condition. Explain why this lack of solution does not contradict the existence theorem.

7. $ty' - y = t^2 \cos t, \quad y(0) = -3$
8. $ty' = 2y - t, \quad y(0) = 2$
9. Show that $y(t) = 0$ and $y(t) = t^3$ are both solutions of the initial value problem $y' = 3y^{2/3}$, where $y(0) = 0$. Explain why this fact does not contradict Theorem 7.16.
10. Show that $y(t) = 0$ and $y(t) = (1/16)t^4$ are both solutions of the initial value problem $y' = ty^{1/2}$, where $y(0) = 0$. Explain why this fact does not contradict Theorem 7.16.

In Exercises 11–16, use a numerical solver to sketch the solution of the given initial value problem.

(i) Where does your solver experience difficulty? Why? Use the image of your solution to estimate the interval of existence.

(ii) For 11–14 only, find an explicit solution; then use your formula to determine the interval of existence. How does it compare with the approximation found in part (i)?

11. $\dfrac{dy}{dt} = \dfrac{t}{y+1}, \quad y(2) = 0$
12. $\dfrac{dy}{dt} = \dfrac{t-2}{y+1}, \quad y(-1) = 1$
13. $\dfrac{dy}{dt} = \dfrac{1}{(t-1)(y+1)}, \quad y(0) = 1$
14. $\dfrac{dy}{dt} = \dfrac{1}{(t+2)(y-3)}, \quad y(0) = 1$
15. $\dfrac{dy}{dt} = \dfrac{2t^2}{(y+3)(y-1)}, \quad y(0) = 0$
16. $\dfrac{dy}{dt} = \dfrac{-t^2}{y(y-5)}, \quad y(0) = 3$

An electric circuit, consisting of a capacitor, resistor, and an electromotive force can be modeled by the differential equation

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t),$$

where R and C are constants (resistance and capacitance) and $q = q(t)$ is the amount of charge on the capacitor at time t. For simplicity in the following analysis, let $R = C = 1$, forming the differential equation $dq/dt + q = E(t)$. In Exercises 17–20, an electromotive force is given in piecewise form, a favorite among engineers. Assume that the initial charge on the capacitor is zero [$q(0) = 0$].

(i) Use a numerical solver to draw a graph of the charge on the capacitor during the time interval [0, 4].

(ii) Find an explicit solution and use the formula to determine the charge on the capacitor at the end of the four-second time period.

17. $E(t) = \begin{cases} 5, & \text{if } 0 < t < 2, \\ 0, & \text{if } t \geq 2 \end{cases}$
18. $E(t) = \begin{cases} 0, & \text{if } 0 < t < 2, \\ 3, & \text{if } t \geq 2 \end{cases}$
19. $E(t) = \begin{cases} 2t, & \text{if } 0 < t < 2, \\ 0, & \text{if } t \geq 2 \end{cases}$
20. $E(t) = \begin{cases} 0, & \text{if } 0 < t < 2, \\ t, & \text{if } t \geq 2 \end{cases}$
21. Consider the initial value problem

$$y' = 3y^{2/3}, \quad y(0) = 0.$$

It is not difficult to construct an infinite number of solutions. Consider

$$y(t) = \begin{cases} 0, & \text{if } t \leq t_0, \\ (t-t_0)^3, & \text{if } t > t_0, \end{cases}$$

where t_0 is any positive number. It is easy to calculate the derivative of $y(t)$, when $t \neq t_0$,

$$y'(t) = \begin{cases} 0, & \text{if } t < t_0, \\ 3(t-t_0)^2, & \text{if } t > t_0, \end{cases}$$

but the derivative at t_0 remains uncertain.

(a) Evaluate both

$$y'(t_0^+) = \lim_{t \searrow t_0} \frac{y(t) - y(t_0)}{t - t_0}$$

and

$$y'(t_0^-) = \lim_{t \nearrow t_0} \frac{y(t) - y(t_0)}{t - t_0},$$

showing that

$$y'(t) = \begin{cases} 0, & \text{if } t \le t_0, \\ 3(t - t_0)^2, & \text{if } t > t_0. \end{cases}$$

(b) Finally, show that $y(t)$ is a solution of 21. Why doesn't this example contradict Theorem 7.16?

22. Consider again the "solution" of equation (7.11) in Example 7.10,

$$y(t) = \begin{cases} 3e^{-2t}, & \text{for } t < 1, \\ 5/2 + (3 - 5e^2/2)e^{-2t} & \text{for } t \ge 1. \end{cases}$$

(a) Follow the lead in Exercise 21 to calculate the derivative of $y(t)$.

(b) In the sense of the definition of solution in Section 2.1, is $y(t)$ a solution of (7.11)? Why or why not?

(c) Show that $y(t)$ satisfies equation (7.11) for all t except $t = 1$.

23. Show that

$$y(t) = \begin{cases} 0, & \text{for } t < 0, \\ t^4 & \text{for } t \ge 0 \end{cases}$$

is a solution of the initial value problem $ty' = 4y$, where $y(0) = 0$, in the sense of Definition 1.15 from Section 2.1. Find a second solution and explain why this lack of uniqueness does not contradict Theorem 7.16.

24. Uniqueness is not just an abstraction designed to please theoretical mathematicians. For example, consider a cylindrical drum filled with water. A circular drain is opened at the bottom of the drum and the water is allowed to pour out. Imagine that you come upon the scene and witness an empty drum. You have no idea how long the drum has been empty. Is it possible for you to determine when the drum was full?

(a) Using physical intuition only, sketch several possible graphs of the height of the water in the drum versus time. Be sure to mark the time that you appeared on the scene on your graph.

(b) It is reasonable to expect that the speed at which the water leaves through the drain depends upon the height of the water in the drum. Indeed, Torricelli's law predicts that this speed is related to the height by the formula $v^2 = 2gh$, where g is the acceleration due to gravity near the surface of the earth. Let A and a represent the area of a cross section of the drum and drain, respectively. Argue that $A\,\Delta h = av\,\Delta t$, and in the limit, $A\,dh/dt = av$. Show that $dh/dt = -(a/A)\sqrt{2gh}$.

(c) By introducing the dimensionless variables $w = \alpha h$ and $s = \beta t$ and then choosing parameters

$$\alpha = \frac{1}{h_0} \quad \text{and} \quad \beta = \left(\frac{a}{A}\right)\sqrt{\frac{2g}{h_0}},$$

where h_0 represents the height of a full tank, show that the equation $dh/dt = -(a/A)\sqrt{2gh}$ becomes $dw/ds = -\sqrt{w}$. Note that when $w = 0$, the tank is empty, and when $w = 1$, the tank is full.

(d) You come along at time $s = s_0$ and note that the tank is empty. Show that the initial value problem, $dw/ds = -\sqrt{w}$, where $w(s_0) = 0$, has an infinite number of solutions. Why doesn't this fact contradict the uniqueness theorem? *Hint:* The equation is separable and the graphs you drew in part (a) should provide the necessary hint on how to proceed.

25. Is it possible to find a function $f(t, x)$ that is continuous and has continuous partial derivatives such that the functions $x_1(t) = t$ and $x_2(t) = \sin t$ are both solutions to $x' = f(t, x)$ near $t = 0$?

26. Is it possible to find a function $f(t, x)$ that is continuous and has continuous partial derivatives such that the functions $x_1(t) = \cos t$ and $x_2(t) = 1 - \sin t$ are both solutions to $x' = f(t, x)$ near $t = \pi/2$?

27. Suppose that x is a solution to the initial value problem

$$x' = x\cos^2 t \quad \text{and} \quad x(0) = 1.$$

Show that $x(t) > 0$ for all t for which x is defined.

28. Suppose that y is a solution to the initial value problem

$$y' = (y - 3)e^{\cos(ty)} \quad \text{and} \quad y(1) = 1.$$

Show that $y(t) < 3$ for all t for which y is defined.

29. Suppose that y is a solution to the initial value problem

$$y' = (y^2 - 1)e^{ty} \quad \text{and} \quad y(1) = 0.$$

Show that $-1 < y(t) < 1$ for all t for which y is defined.

30. Suppose that x is a solution to the initial value problem

$$x' = \frac{x^3 - x}{1 + t^2x^2} \quad \text{and} \quad x(0) = 1/2.$$

Show that $0 < x(t) < 1$ for all t for which x is defined.

31. Suppose that x is a solution to the initial value problem

$$x' = x - t^2 + 2t \quad \text{and} \quad x(0) = 1.$$

Show that $x(t) > t^2$ for all t for which x is defined.

32. Suppose that y is a solution to the initial value problem

$$y' = y^2 - \cos^2 t - \sin t \quad \text{and} \quad y(0) = 2.$$

Show that $y(t) > \cos t$ for all t for which y is defined.