HW 4

§ 2.5 1

§ 3.1 4, 12

§ 2.8 17 ((a), (b))

§ 7.1 40, 45

Read Assignment: § 3.2. Models and the Real World.

Equation (5.5) is separable, so we can separate the variables, integrate, and solve for x, finding that

$$x(t) = C_1 e^{-t/20}.$$

The initial condition $x(0) = 20$ yields $C_1 = 20$, so

$$x(t) = 20e^{-t/20}. \tag{5.7}$$

We now substitute equation (5.7) into equation (5.6) and simplify to obtain the linear equation

$$\frac{dy}{dt} = e^{-t/20} - \frac{1}{40}y. \tag{5.8}$$

Solving in the usual way, we get the general solution

$$y(t) = -40e^{-t/20} + C_2 e^{-t/40}.$$

The initial condition $y(0) = 40$ yields $C_2 = 80$ and

$$y(t) = -40e^{-t/20} + 80e^{-t/40}. \tag{5.9}$$

The solutions in (5.7) and (5.9) are plotted in Figure 6.

Finally, we can use equation (5.9) to find the salt content in tank B at $t =$ 1 min = 60 seconds, finding that

$$y(60) = -40e^{-(60)/20} + 80e^{-(60)/40} \approx 15.9 \text{ lb.}$$

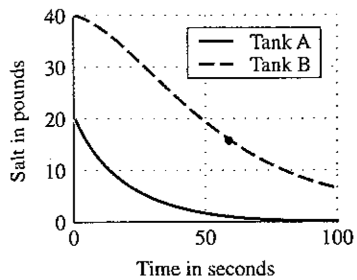

Figure 6. The amount of salt in the tanks in Example 5.4.

EXERCISES

1. A tank contains 100 gal of pure water. At time zero, a sugar-water solution containing 0.2 lb of sugar per gal enters the tank at a rate of 3 gal per minute. Simultaneously, a drain is opened at the bottom of the tank allowing the sugar solution to leave the tank at 3 gal per minute. Assume that the solution in the tank is kept perfectly mixed at all times.
   (a) What will be the sugar content in the tank after 20 minutes?
   (b) How long will it take the sugar content in the tank to reach 15 lb?
   (c) What will be the eventual sugar content in the tank?

2. A tank initially contains 50 gal of sugar water having a concentration of 2 lb of sugar for each gal of water. At time zero, pure water begins pouring into the tank at a rate of 2 gal per minute. Simultaneously, a drain is opened at the bottom of the tank so that the volume of sugar-water solution in the tank remains constant.
   (a) How much sugar is in the tank after 10 minutes?
   (b) How long will it take the sugar content in the tank to dip below 20 lb?
   (c) What will be the eventual sugar content in the tank?

3. A tank initially contains 100 gal of water in which is dissolved 2 lb of salt. The salt-water solution containing 1 lb of salt for every 4 gal of solution enters the tank at a rate of 5 gal per minute. The solution leaves the tank at the same rate, allowing for a constant solution volume in the tank.
   (a) Use an analytic method to determine the eventual salt content in the tank.
   (b) Use a numerical solver to determine the eventual salt content in the tank and compare your approximation with the analytical solution found in part (a).

4. A tank contains 500 gal of a salt-water solution containing 0.05 lb of salt per gallon of water. Pure water is poured into the tank and a drain at the bottom of the tank is adjusted so as to keep the volume of solution in the tank constant. At what rate (gal/min) should the water be poured into the tank to lower the salt concentration to 0.01 lb/gal of water in under one hour?

5. A 50-gal tank initially contains 20 gal of pure water. Salt-water solution containing 0.5 lb of salt for each gallon of water begins entering the tank at a rate of 4 gal/min. Simultaneously, a drain is opened at the bottom of the tank, allowing the salt-water solution to leave the tank at a rate of 2 gal/min. What is the salt content (lb) in the tank at the precise moment that the tank is full of salt-water solution?

6. A tank initially contains 100 gal of a salt-water solution containing 0.05 lb of salt for each gallon of water. At time zero, pure water is poured into the tank at a rate of 3 gal per minute. Simultaneously, a drain is opened at the bottom of the tank that allows the salt-water solution to leave the tank at a rate of 2 gal per minute. What will be the salt

Sensitivity to initial conditions is the idea behind the theory of chaos, which has developed over the past 25 years. In chaotic situations, solutions are sensitive to initial conditions for a large set of possible initial conditions. In the situations we have examined, the sensitivity occurs only at a few isolated points. Such equations do not give rise to truly chaotic behavior.

EXERCISES

Sensitivity to initial conditions is well illustrated by a little target practice with your numerical solver. In Exercises 1–12, you are given a differential equation $x' = f(t, x)$ and a "target." In each case, enter the equation into your numerical solver, and then experiment with initial conditions at the given value of t_0 until the solution of $x' = f(t, x)$, with $x(t_0, x_0)$, "hits" the given target.

We will use the simple linear equation, $x' = x - t$ in Exercises 1–4. The initial conditions are at $t_0 = 0$. The target is

1. $(3, 0)$ **2.** $(4, 0)$

3. $(5, 0)$ **4.** $(6, 0)$

In Exercises 5–8, we use the slightly more complicated nonlinear equation, $x' = x^2 - t$. Again the initial conditions are at $t_0 = 0$. The target is

5. $(3, 0)$ **6.** $(4, 0)$

7. $(5, 0)$ **8.** $(6, 0)$

For Exercises 9–12, we use the equation in Example 8.8, $x' = x \sin x + t$. Again the initial conditions are at $t_0 = 0$. The target is

9. $(3, 0)$ **10.** $(4, 0)$

11. $(5, 0)$ **12.** $(6, 0)$

13. This exercise addresses a very common instance of a motion that is sensitive to initial conditions. Flip a coin with your thumb and forefinger, and let the coin land on a pillow. The motion of the coin is governed by a system of ordinary differential equations. It is not immediately important what that system is. It is only important to realize that the motion is governed entirely by the initial conditions (i.e., the upward velocity of the coin and the rotational energy imparted to it when it is flipped). If the motion were not sensitive to initial conditions, it would be possible to learn how to flip 10 heads in a row. Try to learn how to do this, and report the longest chain of heads you are able to achieve. The flipping of a coin is often considered to have a random outcome. In fact, the result is determined by the initial conditions. It is the sensitivity of the result to the initial conditions that gives the appearance of randomness.

14. Let's plot the error bounds shown in Figure 1. First, solve $x' = (x - 1)\cos t$, $x(0) = 0$, and plot the solution over the interval $[-4, 4]$. Next, as we saw in Example 8.1, if $y(t)$ is a second solution with $|x(0) - y(0)| \le 0.1$, then the inequality (8.3) becomes $|x(t) - y(t)| \le 0.1e^{|t|}$. Solve this inequality for $x(t)$, placing your final answer in the form $e_L(t) \le x(t) \le e_H(t)$. Then add the graphs of $e_L(t)$ and $e_H(t)$ to your plot. How can you use Theorem 7.16 to show that no solution starting with initial condition $|x(0) - y(0)| \le 0.1$ has any chance of rising as far as indicated by $e_H(t)$?

15. Draw the error bounds shown in Figure 2. See Exercise 14 for assistance.

16. Consider the equation $x' = (x - 1)\cos t$.

(a) Let $x(t)$ and $y(t)$ be two solutions. What is the upper bound on the separation $|x(t) - y(t)|$ predicted by Theorem 7.15?

(b) Find the solution $x(t)$ with initial value $x(0) = 0$, and the solution $y(t)$ with initial value $y(0) = 1/10$. Does the separation $x(t) - y(t)$ satisfy the inequality found in part (a)?

(c) Are there any values of t where the separation achieves the maximum predicted?

17. Consider $x' + 2x = \sin t$.

(a) Let $x(t)$ and $y(t)$ be two solutions. What is the upper bound on the separation $|x(t) - y(t)|$ predicted by Theorem 7.15?

(b) Find the solution $x(t)$ with initial value $x(0) = -1/5$, and the solution $y(t)$ with initial value $y(0) = -3/10$. Does the separation $x(t) - y(t)$ satisfy the inequality found in part (a)?

(c) Are there any values of t where the separation achieves the maximum predicted?

18. Let $x_1(t)$ and $x_2(t)$ be solutions of $x' = x^2 - t$ having initial conditions $x_1(0) = 0$ and $x_2(0) = 3/4$. Use Theorem 7.15 to determine an upper bound for $|x_1(t) - x_2(t)|$, as long as the solutions $x_1(t)$ and $x_2(t)$ remain inside the rectangle defined by $R = \{(t, x) : -1 \le t \le 1, -2 \le x \le 2\}$. Use your numerical solver to draw the solutions $x_1(t)$ and $x_2(t)$, restricted to the rectangular region R. Estimate $\max_R |x_1(t) - x_2(t)|$ and compare with the estimated upper bound.

The function P in equation (1.15) has three parameters, P_0, r, and K. The task is to find the values of these parameters that minimize (1.21) or (1.22). This is not a task to be done using a pencil and paper. It is much too difficult. However, it can be done quite easily on a computer if the required programs are available. In either case, the result is called a ***least squares approximation***. Unlike the situation in Example 2, this is a nonlinear least squares problem, which is usually an order of magnitude more difficult than the linear least squares problem found there.[5]

In the case at hand, we used a computer to minimize (1.22), our calculations revealed that the minimum occurred with $P_0 = 9.5998$, $r = 0.5391$, and $K = 665.0000$. The colored curve in Figure 10 is the logistic curve with these parameters. If you have a calculator that will do a logistics regression, you should try to replicate this result.[6] ●

The fit shown by the colored curve in Figure 10 is quite good. Many experiments similar to the one described in Example 1 have shown that the logistic model does a very good job of describing population growth of populations under very carefully controlled conditions, such as those found in a well-run laboratory.

EXERCISES (§ 3.1)

1. A biologist starts with 100 cells in a culture. After 24 hours, he counts 300. Assuming a Malthusian model, what is the reproduction rate? What will be the number of cells at the end of 5 days?

2. A biologist prepares a culture. After 1 day of growth, the biologist counts 1000 cells. After 2 days of growth, he counts 3000. Assuming a Malthusian model, what is the reproduction rate and how many cells were present initially?

3. A population of bacteria is growing according to the Malthusian model. If the population triples in 10 hours, what is the reproduction rate? How often does the population double itself?

4. A population of bacteria, growing according to the Malthusian model, doubles itself in 10 days. If there are 1000 bacteria present initially, how long will it take the population to reach 10,000?

5. A certain bacterium, given plenty of nutrient and room, is known to grow according to the Malthusian model with reproductive rate r. Suppose that the biologist working with the culture harvests the bacteria at a constant rate of h bacteria per hour. Use qualitative analysis to discuss the fate of the culture.

6. A certain bacterium is known to grow according to the Malthusian model, doubling itself every 8 hours. If a biologist starts with a culture containing 20,000 bacteria, then harvests the culture at a constant rate of 2000 bacteria per hour, how long until the culture is depleted? What would happen in the same time span if the initial culture contained 25,000 bacteria?

7. A certain bacterium is known to grow according to the Malthusian model, doubling itself every 4 hours. If a biologist starts with a culture of 10,000 bacteria, at what minimal rate does he need to harvest the culture so that it won't overwhelm the container with bacteria?

8. A biologist grows a culture of fruit flies in a very large enclosure with substantial nutrients available. The following table contains the data on the numbers for each of the first ten days.

Day	Number of flies	Day	Number of flies
0	10	6	55
1	14	7	72
2	19	8	85
3	24	9	123
4	28	10	136
5	38		

[5] Some calculators will do a "logistic regression." However, in at least one case this routine fits a function of the form

$$f(t) = \frac{a}{1 + be^{ct}} + d$$

to the data. Such a routine will not solve the problem we are dealing with. What is needed is a routine that fits a function of the form

$$f(t) = \frac{a}{1 + be^{ct}}$$

to the data. The method used probably changes from calculator to calculator, so you should check it out.

[6] To learn more about fitting the logistic equation, see "Fitting a Logistic Curve to Data," Fabio Cavallini, *College Mathematics Journal*, Vol. 24, Num. 3, pp. 247–253.

(d) Finally, use the change of variables $\omega = \alpha P$ and $s = \beta t$, your parameters α and β found in part (b), and the initial condition $P(t_0) = P_0$ to show that equation (1.32) is equivalent to the solution given in equation (1.15).

12. A population, obeying the logistic equation, begins with 1000 bacteria, then doubles itself in 10 hours. The population is observed eventually to stabilize at 20,000 bacteria. Find the number of bacteria present after 25 hours and the time it takes the population to reach one-half of its carrying capacity.

13. A population is observed to obey the logistic equation with eventual population 20,000. The initial population is 1000, and 8 hours later, the observed population is 1200. Find the reproductive rate and the time required for the population to reach 75% of its carrying capacity.

14. In *The Biology of Population Growth*, published in 1925, the biologist Raymond Pearl reported the data shown in the following table for the growth of a population of fruit flies.

Day	Number of flies	Day	Number of flies
0	6	18	163
3	10	21	226
7	21	24	265
9	52	28	282
12	67	32	319
15	104		

(a) Notice that data were not collected systematically. However, data were collected on days 9 and 18. Use the method of Example 1.20 to estimate the natural reproductive rate and the carrying capacity for a logistic model.

(b) If you have a computer program with a least squares program, use it with all of the data in the table to estimate the natural reproductive rate and the carrying capacity for a logistic model.

(c) Is the logistic model a good one for this data?

15. G. F. Gause, in his *Struggle for Existence*, simply estimated the carrying capacity of a population from the graph of his data. Plot the data shown in the following table, and use your plot to obtain an estimate of the carrying capacity.

Day	Quantity	Day	Quantity
0	100	80	8587
20	476	100	9679
40	1986	120	9933
60	5510	140	9986

You now know the carrying capacity and the initial population. You can use any other point in the table to determine the reproduction rate r. Do so, and then superimpose the resulting logistic curve on your data plot for comparison. Can you see any problems that could occur with this method?

16. Consider a lake that is stocked with walleye pike and that the population of pike is governed by the logistic equation

$$P' = 0.1P(1 - P/10),$$

where time is measured in days and P in thousands of fish. Suppose that fishing is started in this lake and that 100 fish are removed each day.

(a) Modify the logistic model to account for the fishing.

(b) Find and classify the equilibrium points for your model.

(c) Use qualitative analysis to completely discuss the fate of the fish population with this model. In particular, if the initial fish population is 1000, what happens to the fish as time passes? What will happen to an initial population having 2000 fish?

17. A biologist develops a culture that obeys the modified logistic equation

$$P' = 0.38p\left(1 - \frac{P}{1000}\right) - h(t),$$

where the "harvesting" is defined by the piecewise function

$$h(t) = \begin{cases} 200, & \text{if } t < 3, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Use a numerical solver to plot solution trajectories for initial bacterial populations ranging between 0 and 1000. You'll note that in some cases, the population "recovers," but in others, the bacterial count goes to zero. Determine experimentally the critical initial population that separates these two behaviors.

(b) Use an analytic method to determine the exact value of the "critical" initial population found in part (a). Justify your answer.

18. A population, left alone, obeys the logistic law with an initial population of 1000 doubling itself in about 2.3 hours. It is known that the environment can sustain approximately 10,000 individuals. Harvesting is introduced into this environment, with 1500 individuals removed per hour, but only during the last 4 hours of a 24-hour day. Suppose that the population numbers 6000 at the beginning of the day. Use a numerical solver to sketch a graph of the population over the course of the next three days. Find approximately the size of the population at the end of each day.

19. Consider the same lake as in the Exercise 16, but suppose that the fishing is done for a fixed time every day, with the result that 1% of the fish are caught each day.

(a) Modify the logistic model to account for the fishing.

(b) Find and classify the equilibrium points for your model.

(c) If the initial fish population is 1000, what happens to the fish as time passes?

satisfy the given identity.

14. $A + B = B + A$

15. $(A+B)+C = A+(B+C)$

16. $(AB)C = A(BC)$

17. $A(B + C) = AB + AC$

18. $(A + B)C = AC + BC$

19. $(\alpha\beta)A = \alpha(\beta A)$

20. $\alpha(AB) = (\alpha A)B = A(\alpha B)$

21. $(\alpha + \beta)A = \alpha A + \beta A$

22. $\alpha(A + B) = \alpha A + \alpha B$

In Exercises 23–26, show that the scalar and matrices

$$\alpha = -3, \qquad A = \begin{pmatrix} -2 & 4 \\ 4 & 0 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 0 & -5 \\ -2 & 3 \end{pmatrix}$$

satisfy the given identity.

23. $(A^T)^T = A$

24. $(\alpha A)^T = \alpha A^T$

25. $(A + B)^T = A^T + B^T$

26. $(AB)^T = B^T A^T$

We will use the following vectors and matrices in Exercises 27–46.

$$\mathbf{x}_1 = \begin{pmatrix} 9 \\ 5 \end{pmatrix}, \ \mathbf{x}_2 = \begin{pmatrix} -6 \\ -1 \end{pmatrix}, \ \mathbf{x}_3 = \begin{pmatrix} -1 \\ 8 \end{pmatrix}, \ \mathbf{x}_4 = \begin{pmatrix} 7 \\ -9 \end{pmatrix},$$

$$\mathbf{v}_1 = \begin{pmatrix} 10 \\ -5 \\ 3 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 0 \\ 8 \\ 6 \end{pmatrix}, \ \mathbf{v}_3 = \begin{pmatrix} 0 \\ -9 \\ 7 \end{pmatrix}, \ \mathbf{v}_4 = \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix},$$

$$\mathbf{y} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \ \mathbf{z} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \ \mathbf{w} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}, \ \mathbf{u} = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix},$$

$$A = \begin{pmatrix} 9 & -6 \\ 5 & -1 \end{pmatrix}, \ B = \begin{pmatrix} -6 & -1 & 7 \\ -1 & 8 & -9 \end{pmatrix},$$

$$C = \begin{pmatrix} 10 & 0 & -1 \\ -5 & 8 & 3 \\ 3 & 6 & 6 \end{pmatrix}, \ D = \begin{pmatrix} 10 & 0 & 0 & -1 \\ -5 & 8 & -9 & 3 \\ 3 & 6 & 7 & 6 \end{pmatrix}$$

For Exercises 27–32,

(i) Simplify the indicated linear combination.

(ii) Write the indicated linear combination as a matrix product.

(iii) Compute the matrix product found in (ii) using the method in (1.14) and compare your result with the linear combination computed in (i).

27. $2\mathbf{x}_1 + 3\mathbf{x}_2$

28. $-\mathbf{x}_3 + 5\mathbf{x}_4$

29. $4\mathbf{x}_2 - 7\mathbf{x}_4 - 3\mathbf{x}_1$

30. $-2\mathbf{v}_2 + 4\mathbf{v}_3 + 5\mathbf{x}_4$

31. $4\mathbf{v}_1 - 3\mathbf{v}_4 + 3\mathbf{v}_2 + 4\mathbf{v}_3$

32. $\mathbf{v}_2 - 5\mathbf{v}_1 - 3\mathbf{v}_4 - 2\mathbf{v}_3$

For Exercises 33–40, compute the indicated matrix product.

33. $A\mathbf{y}$ **34.** $A\mathbf{x}_3$ **35.** $B\mathbf{z}$ **36.** $B\mathbf{w}$

37. $C\mathbf{z}$ **38.** $C\mathbf{w}$ **39.** $C\mathbf{v}_3$ **40.** $D\mathbf{u}$

41. What is the transpose of $\mathbf{y}$?

42. What is the transpose of $\mathbf{u}$?

43. What is the transpose of A?

44. What is the transpose of B?

45. What is the transpose of C?

46. What is the transpose of D?

For Exercises 47–54, write the indicated system as a matrix equation.

47. $3x + 4y = 7$

48. $-x + 4y = 3$

49. $\begin{aligned} 3x + 4y &= 7 \\ -x + 3y &= 2 \end{aligned}$

50. $\begin{aligned} x - 3y &= 5 \\ -2x + 3y &= -2 \end{aligned}$

51. $\begin{aligned} -x_1 + x_3 &= 0 \\ 2x_1 + 3x_2 &= 3 \end{aligned}$

52. $\begin{aligned} x_1 + 3x_2 - x_3 &= 2 \\ -2x_1 + 3x_2 - 2x_3 &= -3 \end{aligned}$

53. $\begin{aligned} -x_1 + x_3 &= 0 \\ 2x_1 + 3x_2 &= 3 \\ x_2 - x_3 &= 4 \end{aligned}$

54. $\begin{aligned} x_1 + 3x_2 - x_3 &= 2 \\ -2x_1 + 3x_2 - 2x_3 &= -3 \\ 2x_1 \qquad - x_3 &= 0 \end{aligned}$

7.2 Systems of Linear Equations with Two or Three Variables

Before proceeding to the solution of general systems of linear equations, we will look at systems in two or three variables. We will use what we know about vectors and matrices to systematize and simplify our solution method. We will also use what we know about geometry to get a better understanding of solution sets. In the process we will learn how to express solution sets parametrically.

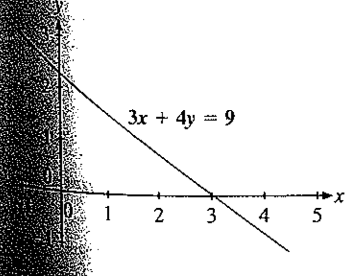

Figure 1. The solution set for a linear equation is a line.

One linear equation in two unknowns

Consider the equation

$$3x + 4y = 9. \tag{2.1}$$

The set of all vectors $(x, y)^T$ that solve this equation will be referred to as the ***solution set*** of (2.1). In this case, the solution set is a line in the plane (see Figure 1).

If we solve (2.1) for x, getting $x = (9 - 4y)/3$, then the vectors in the solution set have the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (9-4y)/3 \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + y \begin{pmatrix} -4/3 \\ 1 \end{pmatrix}. \tag{2.2}$$

20. Adjust the "standard" logistic equation

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

to reflect the fact that a fixed percentage γ of the population is harvested per unit time. Use qualitative analysis to discuss the fate of the population. In your analysis, discuss two particular cases: (1) $\gamma < r$ and (2) $\gamma > r$.

21. In Exercise 20, examine the units of the term γP and explain why the function $Y(\gamma) = \gamma P$ is called the *yield*. Suppose that the harvesting strategy defined in Exercise 20 is kept in place long enough for the population to adjust to its new equilibrium level. What value of γ will maximize the yield at this level? What will be the yield and the new equilibrium level of the population for this value of γ?

3.2 Models and the Real World

Mathematical models are meant to explain what is happening in the real world. It is not enough to derive models from theoretical considerations. It is necessary to check the predictions of our models with what is happening in reality. We did this in the previous section when we looked at the implications of the Malthusian model of population growth. We realized that its prediction of unlimited exponential growth was unrealistic and is only good if the assumption of unlimited resources is satisfied. While this might be true for small populations, it is certainly not true in the long run. We then went back to the drawing board and came up with the logistic model. Our analysis of the experimental data in Example 1.20 showed that in that case, under controlled circumstances, the logistic model worked very well.

We should do the same kind of analysis for the logistic model. If we want to apply the logistic model in new circumstances, it is important to know if these circumstances fit the assumptions behind the logistic model. Let's recall what those assumptions are. First we assumed that the population changed due to births and deaths. We then allowed the birth and death rates to vary with the population in a way that reflected the competition between individuals for limited resources. We did not allow any change of the reproductive rate with respect to time. Now let's look at some other situations.

A Malthusian model of early U.S. population

A number of attempts have been made to model the population of the United States. Pierre-François Verhulst, a Belgian mathematician, argued that the Malthusian model could be used to model rapidly growing populations in environments containing seemingly unlimited resources.

> The United States (in the late eighteenth and early nineteenth centuries) offers just such an example of a rapidly growing population that is expanding as if it had unlimited resources. [Verhulst, Pierre-François. 1845. Recherches mathematiques sur la loi d'accroissement de la population. *Noveaux Mémoires de l'Académie Royale des Sciences et Belles-Letters de Bruxelles* 18: 1–38]

Verhulst used United States census data for the years 1790–1840, provided in Table 1. He used the arithmetic mean to estimate the population in intercensal years, then proceeded to show that the population grew in the geometric progression predicted by the Malthusian model.

If we use regression on the data points in Table 1 as we did in Example 1.8, we get the equation

$$P = 3{,}966{,}000\, e^{0.0294(t-1790)}, \tag{2.1}$$

where the time t is measured in years. The plot of equation (2.1) is superimposed on Verhulst's data in Figure 1. Note that the graph of equation (2.1) is a good fit with the census data.

Table 1 Early population of the United States

YEAR	TIME	POPULATION
1790	0	3,929,827
1800	10	5,305,925
1810	20	7,239,814
1820	30	9,638,151
1830	40	12,866,020
1840	50	17,062,566

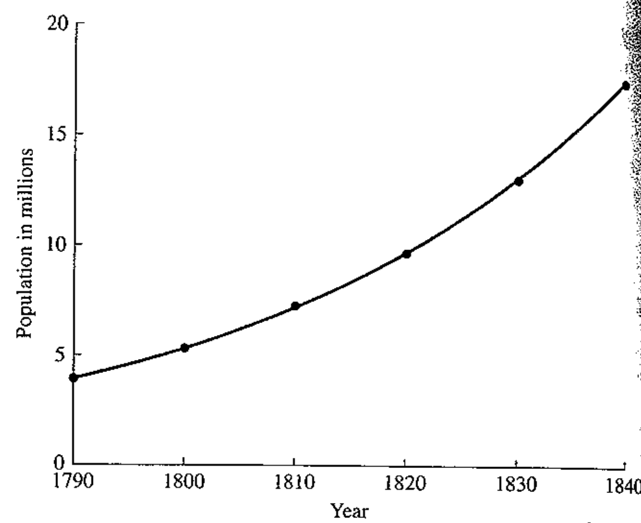

Figure 1. Fitting a Malthusian model to early U.S. population.

Now we come to the main question of this section. Is Verhulst's assertion that this period of U.S. population growth is a good example of "a rapidly growing population that is expanding as if it had unlimited resources" valid? It might be argued that, in addition to births and deaths, there was a third factor affecting the growth of the population, namely immigration. However, in the period between 1790 and 1820, immigration to the United States was rather small in comparison to the population increases that were occurring. Consequently, we can say that Verhulst's use of the Malthusian model was a valid approximation. Indeed, the agreement of (2.1) with the data in Table 1 as shown in Figure 1 is impressive.

Logistic models of U.S. population growth

We have just seen that the early growth of the U.S. population was Malthusian in nature. What about the growth since then?

In 1920, Pearl and Reed used the logistic equation to model the United States population. Their census data, taken from their report to the National Academy of Sciences,[7] came from the Bureau of Census figures. It included the data in Table 2, up to the year 1910. Table 2 comes from the *Statistical Abstract of the United States, 2002*. It shows the population of the United States in thousands.[8] These data are plotted in Figure 2.

By selecting three data points at equally spaced time intervals and using the technique in equations (1.18) and (1.19) of Section 1, Pearl and Reed were able to fit the logistic model to the population data in Table 2 up to 1910. Using the same techniques as before and the populations for 1790, 1850, and 1910, we fit the data

[7] Pearl and Reed, *On the Rate of Growth of the Population of the United States Since 1790 and Its Mathematical Representation*, Proceedings of the National Academy of Sciences, Volume 6, June 15, 1920, Number 6.

[8] The observant reader will notice that the data in Table 2 differs somewhat from that in Table 1. This is simply because the data is from different sources. It frequently happens that different sources report conflicting data. This is sometimes very disconcerting.

Table 2 Population of the United States (in thousands)

Year	Population	Year	Population	Year	Population
1790	3,929	1870	39,818	1940	131,699
1800	5,308	1880	50,156	1950	151,326
1810	7,240	1890	62,948	1960	179,323
1820	9,638	1900	75,995	1970	203,302
1830	12,866	1910	91,972	1980	226,542
1840	17,069	1920	105,711	1990	248,718
1850	23,192	1930	122,755	2000	281,422
1860	31,443				

in Table 2 with the logistic model. The estimates of the parameters are[9]

$$r = 0.0313 \quad \text{and} \quad K = 197{,}274. \tag{2.2}$$

These data, together with $P_0 = 3{,}929{,}000$, are used in equation (1.15), which is

$$P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-r(t-t_0)}}. \tag{2.3}$$

The plot of this function is superimposed on the plot of the U.S. population data in Figure 2. Note the excellent fit.

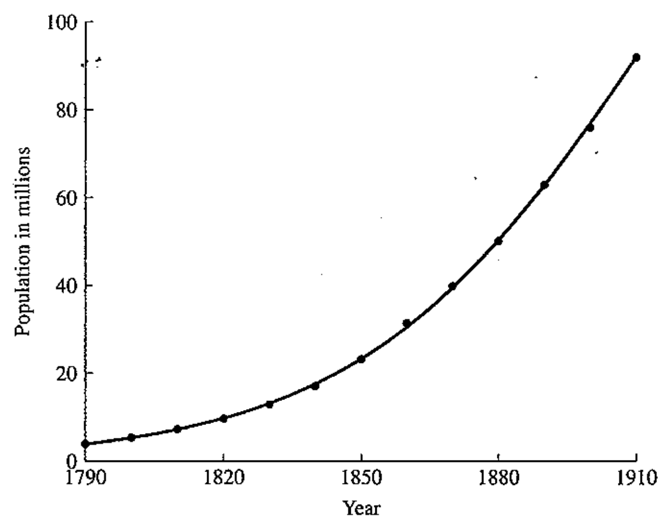

Figure 2. Fitting the logistic model to U.S. population.

Pearl and Reed then used their model to extrapolate the eventual behavior of the U.S. population. Let's check how well their model predicted the population of the United States. In Figure 3, we extend the plot of equation (2.3) with the parameters in (2.2) to the present, and we show the actual data from Table 2.

The model of Pearl and Reed predicted a carrying capacity of 197,274,000 people. However, in 1990 the U.S. population was 248,718,301 people, far beyond that predicted by Pearl and Reed. Thus, we see that using the logistic model to predict the population of the United States failed spectacularly after about 1950. It appears that the logistic model is not very good at predicting the population of the United States.

[9] These computations are a little sensitive, especially to the value of r used to compute K. It is important to keep the actual computed value of r when computing K. Even then, a calculator may not get the value found here due to lack of precision.

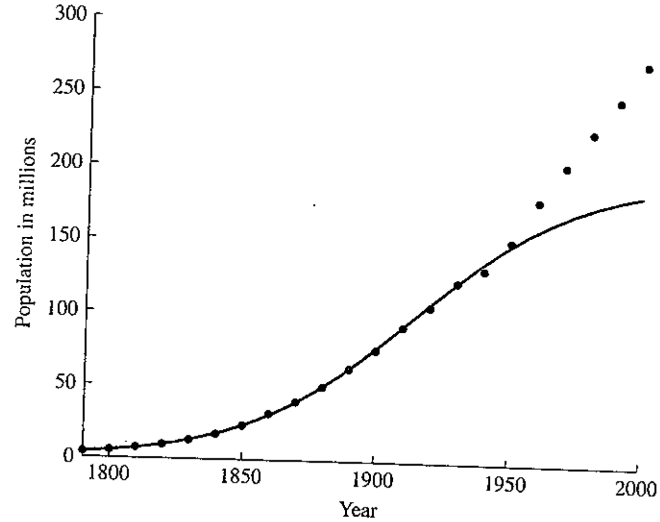

Figure 3. Logistic model projection of U.S. population based on data from 1790, 1850, and 1910, measured against the actual data.

Why is this true? If we think about how the population has grown in the United States, we come to the conclusion that the assumptions of the logistic model are simply not satisfied. For one thing, since about 1830 immigration has been a very important factor. For example, in the first decade of the twentieth century there were 8,795,386 immigrants, while the population increased by 15,977,691. Clearly immigration cannot be ignored. In addition, great improvements in public hygiene and in health care have had a big impact on the birth and death rates. Agricultural technology has increased the production of food by an incredible amount. Birth rates have decreased over the past century because of individual choice—a factor that is unrelated to the availability of resources or to the size of the population.

For these reasons, it should not be a surprise that the logistic model gives such a poor fit with the U.S. population curve. Indeed, it would be highly surprising if the logistic model were a good model for any segment of human population over a very long period of time.

For the total human population of the earth, such a model might be accurate for the period up to about 10,000 years ago, while humans were all hunter/gatherers. However, with the introduction of agriculture, the carrying capacity of the earth increased dramatically. The period from then until the start of the industrial revolution might be another period when a logistic model would be accurate. However, since the start of the industrial revolution changes have occurred at a very high rate, all of which have served to allow the earth to support more people. In addition, there have been changes in the attitudes of people that affect the reproductive rate. It is not likely that a logistic model would be useful for the human population of the earth since that time.

EXERCISES

1. Use the method of Example 1.4 of Section 3.1 and the first two data points in Table 1 to derive equation (2.1).

2. Use equations (1.18) and (1.19) in Section 3.1 together with the population data for 1790, 1850, and 1910 in Table 2 to verify the estimates of the parameters in (2.2).

3. Find in your library or on the Internet the historical census data for one of the United States, or for some other country. Attempt to model the data using the Malthusian model and the logistic model. Critique the effectiveness of the models.