HW 5.

§7.2. 4, 21, 37.

§7.3. 25, 32, 36.

EXERCISES

Use Definition 2.6 to place the solution set of each of the equations in Exercises 1–4 in parametric form. In each exercise, sketch the line represented by the given equation using the parametric technique shown in Figure 2.

1. $3x - 4y = 12$ **2.** $x + 2y = 2$

3. $2x - y = 2$ **4.** $2x + 3y = 6$

For the systems in Exercises 5–8, perform each of the following tasks.

(i) Sketch the system of lines on graph paper and estimate the solution by approximating the point of intersection.

(ii) Set up an augmented matrix for the system. Use elimination to eliminate x in the second equation, then use back-solving to complete the solution of the given system. Compare this exact solution with the approximate solution found in (i).

5. $\begin{aligned} x + 2y &= 6 \\ 2x - 3y &= 6 \end{aligned}$ **6.** $\begin{aligned} x - 2y &= 4 \\ 3x + 2y &= 6 \end{aligned}$

7. $\begin{aligned} x + y &= 3 \\ 2x - 3y &= 6 \end{aligned}$ **8.** $\begin{aligned} x - y &= 4 \\ x + 3y &= 3 \end{aligned}$

For the systems in Exercises 9–12, perform each of the following tasks.

(i) Sketch the system of lines on graph paper and describe the solution set.

(ii) Is the system consistent or inconsistent?

9. $\begin{aligned} x + y &= 3 \\ 2x + 2y &= -4 \end{aligned}$ **10.** $\begin{aligned} x - 2y &= 4 \\ 2x - 4y &= 8 \end{aligned}$

11. $\begin{aligned} x + 2y &= 4 \\ 3x + 6y &= 12 \end{aligned}$ **12.** $\begin{aligned} x - 3y &= 6 \\ 2x - 6y &= 12 \end{aligned}$

Each of the equations in Exercises 13–16 represents a plane in three dimensional space. Place the solution set of the given equation in parametric form as shown in Definition 2.17.

13. $x + 2y - 3z = 12$ **14.** $2x + y - 3z = 6$

15. $3x - 4y + z = 12$ **16.** $2x + 3y + 4z = 6$

The systems in Exercises 17–20 each have two equations with three unknowns. Set up an augmented matrix for the system, eliminate x in the second equation, let the free variable z equal t and then use back-solving to solve for x and y. Place your final answer in parametric form.

17. $\begin{aligned} x + 2y - 3z &= 6 \\ 2x + 5y + 8z &= 40 \end{aligned}$ **18.** $\begin{aligned} x - 3y + 4z &= 12 \\ -3x + 10y + 8z &= 40 \end{aligned}$

19. $\begin{aligned} 2x - 4y + 5z &= 40 \\ -4x + 10y - 4z &= 20 \end{aligned}$ **20.** $\begin{aligned} 3x - 4y + 5z &= 60 \\ -6x + 9y - 4z &= -150 \end{aligned}$

Use the technique shown in Example 2.24 to solve each system in Exercises 21–24.

21. $\begin{aligned} x + y + z &= 3 \\ 2x - y + z &= 4 \\ [illegible] + 2y + 2z &= 6 \end{aligned}$ **22.** $\begin{aligned} x - y + 2z &= 4 \\ -2x + y - z &= 6 \\ 3x + y - 3z &= 7 \end{aligned}$

23. $\begin{aligned} x - 2y + 5z &= 10 \\ -2x + y + z &= 12 \\ 2x - y - z &= 6 \end{aligned}$ **24.** $\begin{aligned} x + y - 2z &= 4 \\ -3x + y - z &= 6 \\ 2x + 6y + 2z &= 4 \end{aligned}$

25. Can a circle in $\mathbf{R}^2$ be the solution set of a system of linear equations?

26. The set S in $\mathbf{R}^2$ contains a point $\mathbf{x}_0$ and every point that is within a distance of 2 from $\mathbf{x}_0$. Is S the solution set of a system of linear equations?

27. Consider $S = \left\{ \begin{pmatrix} t \\ 0 \end{pmatrix} \middle| \, t > 0 \right\}$. Is S the solution set of a system of linear equations?

28. Consider $S = \left\{ \begin{pmatrix} t \\ s \end{pmatrix} \middle| \, t > 0 \right\}$. Is S the solution set of a system of linear equations?

29. Consider the line L in $\mathbf{R}^2$ with the parametric representation

$$\mathbf{y} = t \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

Is L the solution set for a system of linear equations?

30. Consider the line L in $\mathbf{R}^2$ with the parametric representation

$$\mathbf{y} = t \begin{pmatrix} -1 \\ 4 \end{pmatrix}.$$

Is L the solution set for a system of linear equations?

31. Consider the line L in $\mathbf{R}^2$ with the parametric representation

$$\mathbf{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

Is L the solution set for a system of linear equations?

32. Consider the line L in $\mathbf{R}^2$ with the parametric representation

$$\mathbf{y} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 0 \\ -3 \end{pmatrix}.$$

Is L the solution set for a system of linear equations?

Roughly speaking, the ***dimension*** of a solution set is the minimum number of directions we must go from a fixed point in order to reach every point in the solution set.

33. What are the dimensions of the possible solution sets in $\mathbf{R}^2$?

34. What are the dimensions of the possible solution sets in $\mathbf{R}^3$?

35. Argue by analogy to decide the dimensions of the possible solution sets in $\mathbf{R}^4$.

36. Find a parametric representation for the solution set of the equation

$$x_1 + 2x_2 - 2x_3 + x_4 = 2.$$

What is its dimension? For your information, a set such as this is called a ***hyperplane*** in $\mathbf{R}^4$.

37. Find a parametric representation for the solution set of the system of equations

$$\begin{aligned} x_1 + 2x_2 - 2x_3 + x_4 &= 2 \\ x_2 - 3x_3 - x_4 &= 3. \end{aligned}$$

What is its dimension? How would you describe the solution set?

38. Find a parametric representation for the solution set of the system of equations

$$\begin{aligned} x_1 + 2x_2 - 2x_3 + x_4 &= 2 \\ x_2 - 3x_3 - x_4 &= 3 \\ x_3 - x_4 &= 0. \end{aligned}$$

What is its dimension? How would you describe the solution set?

7.3 Solving Systems of Equations

In Section 7.2 we learned how to solve a few systems of linear equations. Examples 2.20 and 2.24 were especially illuminating. In this section we will find a systematic way to solve systems of equations, a way that always works and that we could easily program on a computer. Sophistication is not important here. We are only interested in finding a way that always works. Furthermore, we want a method that finds all solutions when there is more than one.

As we already indicated in Section 7.2, the method will use the augmented matrix. We will eliminate coefficients to put the system into an equivalent form that can be easily solved by the method of back-solving. Our first task will be to describe the form of equations that can be solved easily in this way. We will then describe in more detail how to go about the process of elimination. Finally, we will discover some more useful facts about process of back-solving.

Row echelon form of a matrix — the goal of elimination

Look back at the augmented matrices of the systems which we solved by back-solving in Section 7.2. These are in equations (2.12), (2.22), and (2.28). To describe how these matrices are different from the others, we will introduce some terminology.

DEFINITION 3.1 The *pivot* of a row vector in a matrix is the first nonzero element of that row.

To give us a specific example to talk about, consider the system

$$\begin{aligned} 2x_2 + 4x_3 &= 2 \\ x_1 \qquad - 2x_3 &= -1 \\ -2x_1 + 2x_2 + 8x_3 &= 4. \end{aligned} \tag{3.2}$$

This system can be written as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 0 & 2 & 4 \\ 1 & 0 & -2 \\ -2 & 2 & 8 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}. \tag{3.3}$$

The augmented matrix for the system is

$$M = [A, \mathbf{b}] = \begin{pmatrix} 0 & 2 & 4 & 2 \\ 1 & 0 & -2 & -1 \\ -2 & 2 & 8 & 4 \end{pmatrix}. \tag{3.4}$$

In this matrix the pivots of each row are printed in blue.

 (7.3).

If you have a computer or calculator that will place an augmented matrix in reduced row echelon form, use it to help find the solution of each system $Ay = \mathbf{b}$ given in Exercises 23–36. Otherwise you'll have to do the calculations by hand.

23. $A = \begin{pmatrix} -6 & 8 & 0 \\ 4 & 8 & 8 \\ -2 & 2 & 7 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 20 \\ 7 \end{pmatrix}$

24. $A = \begin{pmatrix} 3 & -3 & 1 \\ 7 & -4 & 5 \\ 4 & -3 & -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$

25. $A = \begin{pmatrix} 4 & 2 & -5 \\ -14 & -8 & 18 \\ -3 & -2 & 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -5 \\ 16 \\ 3 \end{pmatrix}$

26. $A = \begin{pmatrix} -4 & 10 & -6 \\ 0 & -4 & 4 \\ 2 & -10 & 8 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -14 \\ 4 \\ 12 \end{pmatrix}$

27. $A = \begin{pmatrix} -3 & -3 & 1 \\ 8 & 7 & -2 \\ 8 & 6 & -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ -8 \\ -5 \end{pmatrix}$

28. $A = \begin{pmatrix} 5 & 9 & 2 \\ -2 & -3 & -1 \\ 0 & -2 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 8 \\ -2 \\ -4 \end{pmatrix}$

29. $A = \begin{pmatrix} -12 & 12 & -8 \\ -16 & 16 & -10 \\ -3 & 3 & -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -8 \\ -10 \\ -1 \end{pmatrix}$

30. $A = \begin{pmatrix} 0 & 4 & 6 & -7 \\ -4 & 10 & 4 & -8 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$

31. $A = \begin{pmatrix} -5 & -4 & 4 & 3 \\ 7 & 6 & -5 & 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 17 \\ 12 \end{pmatrix}$

32. $A = \begin{pmatrix} 2 & -3 & -2 & 2 \\ -4 & 4 & 0 & 3 \\ 8 & -8 & -1 & -7 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -4 \\ -7 \\ 13 \end{pmatrix}$

33. $A = \begin{pmatrix} -7 & 7 & -8 & -3 \\ 9 & -5 & 8 & -2 \\ 5 & 0 & 2 & 8 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 37 \\ -35 \\ -9 \end{pmatrix}$

34. $A = \begin{pmatrix} -7 & -4 & -5 & -9 \\ 1 & 10 & 7 & 10 \\ -2 & 0 & -3 & 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 31 \\ -18 \\ -2 \end{pmatrix}$

35. $A = \begin{pmatrix} 8 & -6 & 9 & 8 & -1 \\ -9 & 5 & -7 & 9 & 0 \\ 1 & -4 & 1 & -3 & -7 \end{pmatrix}$ and
$\mathbf{b} = \begin{pmatrix} 15 \\ -30 \\ 9 \end{pmatrix}$

36. $A = \begin{pmatrix} -2 & 3 & 6 & -7 & -1 \\ -1 & -6 & 1 & -6 & -8 \\ 0 & -9 & -5 & -1 & 9 \end{pmatrix}$ and
$\mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 16 \end{pmatrix}$

7.4 Homogeneous and Inhomogeneous Systems

In every example of a consistent system we have seen so far, the solution set has been given by a parametric equation.[2] So far these have been points, lines, or planes; however, higher-dimensional sets do arise. In fact, any solution set can be presented parametrically, and in this section we will begin the process of understanding solution sets and their parametric representations better. We will start by looking closely at homogeneous systems. With a better understanding of these, we will study the relationship between the solution set of an inhomogeneous system and that of the related homogeneous system.

It is highly informative to compare our results here to the results for linear differential equations in Chapters 2 and 4.

Homogeneous systems

Remember that a homogeneous system has the form $A\mathbf{x} = \mathbf{0}$, where the right-hand side is $\mathbf{0}$, the zero vector. Since $A\mathbf{0} = \mathbf{0}$, a homogeneous system always has the solution $\mathbf{x} = \mathbf{0}$, and is therefore consistent. Sometimes this is the only solution, but there may be others as well. How can we tell if there are solutions other than $\mathbf{0}$? This question has important application to differential equations. We will call a solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$ that is different from $\mathbf{0}$ a ***nontrivial solution***. To be nontrivial, a vector must have at least one nonzero component.

Let's look at a couple of examples.

[2] This is true even when the solution set was a single point. For example, in Example 2.25 the solution in vector form is $\mathbf{x} = (1, 0, -1)^T$. This is a parametric equation with no parameters.