HW 6:

§ 7.4. 16, 27

§ 7.6. 26

§ 7.7. 37, 38 46

For 26, 46. only calculate the determint of the matrix.

(§7.4)

The matrix A and the solution of $A\mathbf{x} = \mathbf{0}$ follow.

$$A = \begin{pmatrix} 1 & 0 & -4 & 2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{x} = s\begin{pmatrix} 4 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

The vectors $(4, -2, 1, 0)^T$ and $(-2, 3, 0, 1)^T$ are "special" because all solutions of $A\mathbf{x} = \mathbf{0}$ can be written as a linear combination of these two vectors. In Exercises 11–14, perform each of the following tasks.

(i) Use a computer or calculator to place the given matrix A in reduced row echelon form. How many free variables does the reduced row echelon form have?

(ii) Write the solution to $A\mathbf{x} = \mathbf{0}$ in parametric form. How many "special" vectors are there?

11. $A = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 2 & 0 & 0 & -2 \\ 3 & -4 & 4 & 1 \end{pmatrix}$

12. $A = \begin{pmatrix} -3 & -5 & -1 & -5 \\ -2 & -3 & -1 & -3 \\ 0 & 3 & -3 & 3 \end{pmatrix}$

13. $A = \begin{pmatrix} 0 & -2 & -2 & 0 & 2 \\ -3 & 1 & 4 & 0 & -1 \\ 1 & 0 & -1 & 0 & 0 \end{pmatrix}$

14. $A = \begin{pmatrix} -1 & -1 & -2 & 2 & -3 \\ 3 & 3 & 1 & -1 & 4 \\ 0 & 0 & 2 & -2 & 2 \end{pmatrix}$

15. Consider the system

$$\begin{aligned} -x_1 + x_2 + x_3 - x_4 \qquad\quad - x_6 &= 0 \\ x_1 - 4x_2 + 2x_3 + 4x_4 - 3x_5 - 2x_6 &= 0 \\ 3x_1 - 3x_2 - 3x_3 + 3x_4 \qquad\quad + 3x_6 &= 0. \end{aligned}$$

(a) This system must possess at least how many free variables? Answer this question without doing any calculations and explain your reasoning.

(b) Could the system have more free variables than the number proposed in part (a)? Set up and reduce the augmented matrix to find out.

16. Consider the system

$$\begin{aligned} 3x_1 \qquad + 3x_3 - 3x_4 \qquad\quad - 3x_6 + 3x_7 &= 0 \\ -x_1 + 3x_2 + 2x_3 + x_4 + 3x_5 + 4x_6 + 2x_7 &= 0 \\ x_1 - 2x_2 - x_3 - x_4 - 2x_5 - 3x_6 - x_7 &= 0 \\ -x_1 + x_2 \qquad + x_4 + x_5 + 2x_6 \qquad &= 0. \end{aligned}$$

(a) This system must possess at least how many free variables? Answer this question without doing any calculations and explain your reasoning.

(b) Could the system have more free variables than the number proposed in part (a)? Set up and reduce the augmented matrix to find out.

17. Prove part (2) of Proposition 4.21.

In Exercises 18–21, the matrix A and the vector $\mathbf{b}$ of the system $A\mathbf{x} = \mathbf{b}$ are given. Perform each of the following tasks for each exercise.

(i) As in Example 4.16, find the solution to the system in the form $\mathbf{x} = \mathbf{p} + \mathbf{v}$, where $\mathbf{p}$ is a particular solution and $\mathbf{v}$ is given in parametric form.

(ii) Show that the $\mathbf{v}$ is in the nullspace of the given matrix A by showing directly that $A\mathbf{v} = \mathbf{0}$. *Hint*: See Exercises 1 and 2.

18. $A = \begin{pmatrix} 2 & -2 & 1 \\ 1 & -1 & -1 \\ 0 & 0 & 3 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix}$

19. $A = \begin{pmatrix} 5 & -2 & -5 \\ -3 & 0 & 3 \\ 0 & -3 & 0 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$

20. $A = \begin{pmatrix} -3 & 3 & 3 & 0 \\ -4 & 2 & 4 & -2 \\ 1 & 0 & -1 & 1 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$

21. $A = \begin{pmatrix} 2 & 0 & 0 & -4 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

In Exercises 22–25, sketch the nullspace of the given matrix in $\mathbf{R}^2$.

22. $A = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$ **23.** $A = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}$

24. $A = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$ **25.** $A = \begin{pmatrix} 3 & -2 \\ 6 & -4 \end{pmatrix}$

In Exercises 26–29, describe the nullspace of the given matrix, both parametrically and geometrically in $\mathbf{R}^3$.

26. $\begin{pmatrix} 3 & -1 & 0 \\ -1 & 0 & 2 \\ 4 & 0 & -2 \end{pmatrix}$ **27.** $\begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & -2 \\ 2 & 0 & -2 \end{pmatrix}$

28. $\begin{pmatrix} 4 & -4 & -4 \\ -5 & 5 & 5 \\ 3 & -3 & -3 \end{pmatrix}$ **29.** $\begin{pmatrix} 1 & -1 & -1 \\ 2 & -2 & -2 \\ -4 & 4 & 4 \end{pmatrix}$

7.5 Bases of a Subspace

We have discovered that the nullspace of a matrix consists of the set of *all* linear combinations of a few vectors. In Definition 5.1, we will define this as the span of those vectors, and then proceed to study this structure in more detail. We will discover the importance of the concept of linear independence. This will lead us to the notion of a basis. Finally, this will allow us to define the dimension of a nullspace.

 (§7.6)

22. $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

23. $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

24. $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$

25. $A = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

26. $A = \begin{pmatrix} 0 & -3 & -1 & 2 \\ -3 & 0 & 0 & 0 \\ 2 & 1 & -2 & -2 \\ -3 & -1 & 3 & 4 \end{pmatrix}$

27. $A = \begin{pmatrix} 3 & -1 & -3 & -1 \\ 0 & -3 & -4 & 1 \\ -2 & 1 & 2 & 1 \\ 1 & 1 & 3 & -3 \end{pmatrix}$

In Exercises 28–33, without actually solving, which systems have unique solutions? Explain.

28. $x_1 + 2x_2 = 4$
$x_1 - x_2 = 6$

29. $x_1 + 2x_2 = 4$
$2x_1 + 4x_2 = 8$

30. $x_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \end{pmatrix}$

31. $x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

32. $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & -1 \\ 1 & -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

33. $\begin{pmatrix} 1 & 0 & 3 \\ -1 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

34. For which values of x is the matrix

$$\begin{pmatrix} 1 & 3 & -2 \\ 2 & 8 & x \\ 0 & 8 & 5 \end{pmatrix}$$

invertible?

35. List as many properties as you can of an invertible matrix.

36. List as many properties as you can of a nonsingular matrix.

7.7 Determinants

There is a question that is still unanswered. Given a matrix A, is there an easy way to tell if its nullspace is nontrivial? For a square matrix, we found a partial answer in Section 7.6. According to Proposition 6.6, a square matrix has a nontrivial nullspace if and only if it is singular. However, we do not as yet have an easy way to tell if a matrix is singular or nonsingular. Proposition 6.2 tells us that A is nonsingular if and only if when it is transformed into row echelon form, all of the diagonal entries are nonzero, but this is not an adequate answer. A reasonable answer is provided by the determinant.

Let's look at the 2×2 case

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If we assume that $a \neq 0$ and put A into row echelon form with a row operation, we get

$$\begin{pmatrix} a & b \\ 0 & d - bc/a \end{pmatrix}.$$

The diagonal entries will all be nonzero if and only if their product $a \cdot (d - bc/a) = ad - bc \neq 0$. You will recognize that $ad - bc$ is the determinant of A. Thus A is nonsingular if and only if $\det(A) \neq 0$.

Carrying through this calculation for the 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

is tedious, but we again end up with the result that A is nonsingular if and only if $\det(A) \neq 0$. However, now the determinant has the more complicated form

$$\begin{aligned} \det(A) = {} & a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ & + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} + a_{13}a_{21}a_{32}. \end{aligned} \tag{7.1}$$

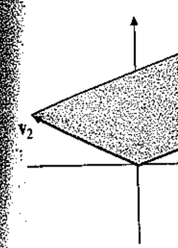

Figure 1. The are[a] ... parallelogram P is ... $|\det([\mathbf{v}_1, \mathbf{v}_2])|$.

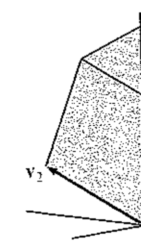

Figure 2. The vo[lume] ... parallelepiped P ... $|\det([\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3])$

upper triangular form. Use hand calculations only. No technology is allowed.

7. $\begin{pmatrix} 3 & 0 & 0 \\ 3 & 6 & 3 \\ -18 & -18 & -9 \end{pmatrix}$
8. $\begin{pmatrix} 5 & 6 & 4 \\ -4 & -9 & -8 \\ 4 & 6 & 5 \end{pmatrix}$

9. $\begin{pmatrix} 1 & 0 & 4 \\ -3 & 3 & -2 \\ 4 & -1 & -2 \end{pmatrix}$
10. $\begin{pmatrix} 1 & 2 & -3 \\ 0 & 6 & -2 \\ -2 & 3 & 2 \end{pmatrix}$

11. $\begin{pmatrix} 2 & -1 & 3 & 4 \\ 0 & 2 & -2 & 0 \\ -1 & 2 & 0 & 0 \\ -1 & 3 & 1 & 2 \end{pmatrix}$
12. $\begin{pmatrix} 3 & -3 & -2 & -1 \\ 2 & 0 & -2 & -1 \\ 1 & -2 & 0 & 0 \\ 4 & -1 & -4 & -2 \end{pmatrix}$

13. Let A be an arbitrary $n \times n$ matrix.
   (a) If the ith row of A is a scalar multiple of the jth row, prove that the determinant of A is zero. State and prove a similar statement about the columns of A.
   (b) Without computing the determinant, explain why each of the following matrices has a zero determinant.

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} -1 & 2 & 0 \\ 3 & 4 & 0 \\ 5 & 2 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 5 & 1 & 2 \end{pmatrix} \quad \begin{pmatrix} -1 & -2 & 3 \\ 1 & 2 & 1 \\ 2 & 4 & 1 \end{pmatrix}$$

14. Let A be an arbitrary $n \times n$ matrix.
   (a) If row i is a linear combination of the preceding rows, prove that the determinant of A is zero. State and prove a similar statement about the columns of A.
   (b) Without computing the determinant, explain why each of the following matrices has a zero determinant.

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 1 \\ 0 & 3 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & 3 \\ -1 & 1 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 3 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 5 \\ -1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

15. Suppose that matrix A is $n \times n$ and is ***lower triangular*** (i.e., $a_{ij} = 0$ if $i < j$). Explain why the determinant is still equal to the product of the diagonal elements. Calculate the determinant of

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 1 & 4 \end{pmatrix}.$$

In Exercises 16–21, expand the matrix by row or column to calculate the determinant.

16. Matrix A in Exercise 7
17. Matrix A in Exercise 8
18. Matrix A in Exercise 9
19. Matrix A in Exercise 10
20. Matrix A in Exercise 11
21. Matrix A in Exercise 12

In Exercises 22–29, calculate the determinant of the given matrix. Determine if the matrix has a nontrivial nullspace, and if it does find a basis for the nullspace. Determine if the column vectors in the matrix are linearly independent.

22. $\begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix}$
23. $\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}$

24. $\begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix}$
25. $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$

26. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
27. $\begin{pmatrix} 1 & -2 & -4 \\ 2 & 1 & 2 \\ 3 & 0 & 0 \end{pmatrix}$

28. $\begin{pmatrix} -1 & 0 & -1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$
29. $\begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & 5 \\ 1 & 0 & -1 \end{pmatrix}$

In Exercises 30–39, find the values of x for which the indicated matrix has a nontrivial nullspace.

30. $\begin{pmatrix} 2 & x \\ 3 & -2 \end{pmatrix}$
31. $\begin{pmatrix} 2 & x \\ x & 3 \end{pmatrix}$

32. $\begin{pmatrix} x & 4 \\ 3 & -2 \end{pmatrix}$
33. $\begin{pmatrix} -1 & x \\ -x & 4 \end{pmatrix}$

34. $\begin{pmatrix} 2-x & 1 \\ 0 & -1-x \end{pmatrix}$
35. $\begin{pmatrix} -1-x & 0 \\ 3 & 2-x \end{pmatrix}$

36. $\begin{pmatrix} -1-x & 5 & 2 \\ 0 & -x & -1 \\ 0 & 6 & -5-x \end{pmatrix}$

37. $\begin{pmatrix} 2-x & 0 & 0 \\ -1 & -x & 2 \\ 0 & -2 & 5-x \end{pmatrix}$

38. $\begin{pmatrix} 2-x & 0 & 1 \\ -3 & -1-x & -1 \\ -2 & 0 & -1-x \end{pmatrix}$

39. $\begin{pmatrix} -1-x & 2 & 2 \\ 0 & -2-x & 0 \\ -1 & 4 & 2-x \end{pmatrix}$

In Exercises 40–49, compute the determinant of the matrix. In each case, decide if there is a nonzero vector in the nullspace.

40. $\begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}$
41. $\begin{pmatrix} -2 & 3 \\ -2 & 4 \end{pmatrix}$

42. $\begin{pmatrix} -1 & -9 & 10 \\ -7 & -19 & 26 \\ -2 & -10 & 12 \end{pmatrix}$
43. $\begin{pmatrix} 1 & 0 & 4 \\ -3 & 3 & -2 \\ 4 & 0 & -2 \end{pmatrix}$

44. $\begin{pmatrix} 1 & 2 & -3 \\ 0 & 6 & -2 \\ -2 & 3 & 2 \end{pmatrix}$
45. $\begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & -2 \\ -1 & 0 & 3 \end{pmatrix}$

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46. $\begin{pmatrix} 10 & -1 & -3 & 9 \\ 3 & 2 & -3 & 3 \\ 3 & 1 & -2 & 3 \\ -10 & 2 & 2 & -9 \end{pmatrix}$

47. $\begin{pmatrix} 3 & 0 & 20 & -8 \\ 2 & 3 & -2 & 0 \\ 6 & 4 & 17 & -8 \\ 16 & 10 & 50 & -23 \end{pmatrix}$

48. $\begin{pmatrix} -120 & 60 & -79 & 52 & 68 & 123 \\ -262 & 162 & -216 & 124 & 184 & 262 \\ -142 & 78 & -100 & 64 & 92 & 142 \\ -262 & 162 & -216 & 124 & 184 & 262 \\ -10 & -4 & 6 & -2 & 2 & 10 \\ -112 & 60 & -79 & 52 & 68 & 115 \end{pmatrix}$

49. $\begin{pmatrix} 556 & 65 & -91 & 52 & 416 & -143 \\ 550 & 60 & -90 & 50 & 410 & -140 \\ -169 & -22 & 26 & -18 & -131 & 38 \\ -96 & -13 & 14 & -7 & -69 & 27 \\ 550 & 60 & -90 & 50 & 410 & -140 \\ 825 & 97 & -137 & 80 & 617 & -211 \end{pmatrix}$

50. Suppose that U is a 4×4 matrix with $\det(U) = -3$.
(a) What is the value of $\det(-2U)$?
(b) What is the value of $\det(U^3)$?
(c) What is the value of $\det(U^{-1})$?

51. Prove or disprove: If A and B are $n \times n$ matrices, then $\det(A + B) = \det(A) + \det(B)$.

52. Two matrices A and B are ***similar matrices*** if there exists a nonsingular matrix S such that $A = S^{-1}BS$.
(a) Prove: If A and B are similar, then $\det(A) = \det(B)$.
(b) Prove: If A and B are similar, then $\det(A - \lambda I) = \det(B - \lambda I)$.

53. Make a list of as many properties as you can find that are equivalent to a matrix having a nonzero determinant.

54. Make a list of as many properties as you can find that are equivalent to a matrix having a zero determinant.

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