HW 7:

§ 7.5 22, 31, 37

§ 7.6 17, 27

§ 8.1 16

PS: Prop 6.6. $Ax=0$ has a nonzero solution iff A is singular.

8. Is the vector $\mathbf{w} = (-7, 22, -25)^T$ in the span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

9. Let $\mathbf{v}_1 = (1, -2)^T$ and $\mathbf{v}_2 = (2, 3)^T$. Show that span$\{\mathbf{v}_1, \mathbf{v}_2\} = \mathbf{R}^2$ by showing that any vector $\mathbf{w} = (w_1, w_2)^T$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. *Note*: Find a specific linear combination (in terms of w_1 and w_2) of $\mathbf{v}_1$ and $\mathbf{v}_2$ that equals $\mathbf{w}$.

10. Let $\mathbf{v}_1 = (0, -1, -2)^T$, $\mathbf{v}_2 = (-2, 1, -4)^T$, and $\mathbf{v}_3 = (-2, -2, 0)^T$. Show that span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbf{R}^3$ by showing that any vector $\mathbf{w} = (w_1, w_2, w_3)^T$ can be written as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. *Note*: Find a specific linear combination (in terms of w_1, w_2, and w_3) of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ that equals $\mathbf{w}$.

In Exercises 11–16, each set of vectors presented is linearly dependent. Use the technique of Example 5.16 to find a nontrivial linear combination of the given vectors that equal the zero vector. Check your solution.

11. $\mathbf{v}_1 = (1, 1, -2)^T$, $\mathbf{v}_2 = (1, 2, 2)^T$, and $\mathbf{v}_3 = (3, 4, -2)^T$

12. $\mathbf{v}_1 = (2, -3, 3)^T$, $\mathbf{v}_2 = (5, -2, 5)^T$, and $\mathbf{v}_3 = (-3, -1, -2)^T$

13. $\mathbf{v}_1 = (1, -2, -2)^T$, $\mathbf{v}_2 = (0, 1, 5)^T$, and $\mathbf{v}_3 = (2, -1, 11)^T$

14. $\mathbf{v}_1 = (-3, -2, -1)^T$, $\mathbf{v}_2 = (2, 0, 2)^T$, and $\mathbf{v}_3 = (7, 2, 5)^T$

15. $\mathbf{v}_1 = (1, 2, -2, 0)^T$, $\mathbf{v}_2 = (2, 0, 2, 3)^T$, $\mathbf{v}_3 = (-2, 4, -8, -6)^T$, and $\mathbf{v}_4 = (6, -4, 12, 12)^T$

16. $\mathbf{v}_1 = (1, 1, 2, -1)^T$, $\mathbf{v}_2 = (3, 1, 0, -1)^T$, $\mathbf{v}_3 = (2, 0, -2, 0)^T$, and $\mathbf{v}_4 = (1, 3, 8, -3)^T$

For each of the sets of vectors in Exercises 17–24, use the technique demonstrated in Examples 5.16 and 5.17 either to show that they are linearly independent or find a nontrivial linear combination that is equal to $\mathbf{0}$.

17. $\mathbf{v}_1 = (1, 2)^T$ and $\mathbf{v}_2 = (-1, 3)^T$

18. $\mathbf{v}_1 = (-2, 3)^T$ and $\mathbf{v}_2 = (2, -6)^T$

19. $\mathbf{v}_1 = (-1, 7, 7)^T$ and $\mathbf{v}_2 = (-3, 7, -4)^T$

20. $\mathbf{v}_1 = (-8, 9, -6)^T$ and $\mathbf{v}_2 = (-2, 0, 7)^T$

21. $\mathbf{v}_1 = (-1, 7, 7)^T$, $\mathbf{v}_2 = (-3, 7, -4)^T$, and $\mathbf{v}_3 = (-4, -14, 23)^T$

22. $\mathbf{v}_1 = (-8, 9, -6)^T$, $\mathbf{v}_2 = (-2, 0, 7)^T$, and $\mathbf{v}_3 = (8, -18, 40)^T$

23. $\mathbf{v}_1 = (-1, 7, 7)^T$, $\mathbf{v}_2 = (-3, 8, -4)^T$, and $\mathbf{v}_3 = (-4, -14, 23)^T$

24. $\mathbf{v}_1 = (-8, 9, -6)^T$, $\mathbf{v}_2 = (-2, -1, 7)^T$, and $\mathbf{v}_3 = (8, -18, 40)^T$

Use the technique of Example 5.21 to find a basis for the nullspace of the matrices given in Exercises 25–32.

25. $\begin{pmatrix} 2 & -1 \end{pmatrix}$

26. $\begin{pmatrix} -3 & 5 \end{pmatrix}$

27. $\begin{pmatrix} 4 & 4 \\ -2 & -2 \end{pmatrix}$

28. $\begin{pmatrix} 4 & 4 \\ -2 & -1 \end{pmatrix}$

29. $\begin{pmatrix} 1 & 1 & 1 \\ -5 & -2 & -5 \\ 1 & 0 & 1 \end{pmatrix}$

30. $\begin{pmatrix} -3 & 8 & -11 \\ -4 & 10 & -14 \\ -2 & 5 & -7 \end{pmatrix}$

31. $\begin{pmatrix} 2 & -1 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 1 & 1 & 3 & 2 \\ -3 & 3 & 3 & 0 \end{pmatrix}$

32. $\begin{pmatrix} -8 & 14 & -24 & 14 \\ 4 & -10 & 18 & -10 \\ 4 & -8 & 14 & -8 \\ -2 & 5 & -9 & 5 \end{pmatrix}$

Use the technique shown in Example 5.22 to help find a basis for the span of the sets of vectors in Exercises 33–40. What is the dimension of the span?

33. The set in Exercise 17

34. The set in Exercise 18

35. The set in Exercise 19

36. The set in Exercise 20

37. The set in Exercise 21

38. The set in Exercise 22

39. The set in Exercise 23

40. The set in Exercise 24

In Exercises 41–44, give a geometric description of the span of the given vectors in the given space.

41. $\mathbf{v}_1 = (1, -2)^T$, $\mathbf{v}_2 = (2, -4)^T$, in $\mathbf{R}^2$

42. $\mathbf{v}_1 = (2, -2)^T$, $\mathbf{v}_2 = (2, -4)^T$, in $\mathbf{R}^2$

43. $\mathbf{v}_1 = (-1, 3, 3)^T$, $\mathbf{v}_2 = (-3, -2, 2)^T$, and $\mathbf{v}_3 = (4, -1, -5)^T$ in $\mathbf{R}^3$

44. $\mathbf{v}_1 = (1, 2, 3)^T$, $\mathbf{v}_2 = (2, 4, 6)^T$, and $\mathbf{v}_3 = (-5, -10, -15)^T$ in $\mathbf{R}^3$

7.6 Square Matrices

In this section we will focus on square matrices. These arise as the coefficient matrices for systems of n equations in n unknowns. For purposes of application to differential equations square matrices are the most important.

We will discuss two properties of square matrices. We define a nonsingular matrix in Definition 6.1 and an invertible matrix in Definition 6.10. These definitions are quite different, yet one of the important results of this section is that they are equivalent (Proposition 6.11).

We will examine these properties of matrices using the reduction to row echelon form described in Section 7.3. The key observation is that any system of equations can be reduced to an equivalent system that is in row echelon form.

$$\begin{matrix} 0 & -1 & -3 & -3 & 0 & 0 & 0 & 1 \end{matrix}$$

The reduced row echelon form of M is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 & 4 & 3 & 1 \\ 0 & 1 & 0 & 0 & -3 & 6 & 6 & 5 \\ 0 & 0 & 1 & 0 & 3 & -5 & -5 & -5 \\ 0 & 0 & 0 & 1 & -2 & 3 & 3 & 3 \end{pmatrix}.$$

Hence

$$A^{-1} = \begin{pmatrix} -2 & 4 & 3 & 1 \\ -3 & 6 & 6 & 5 \\ 3 & -5 & -5 & -5 \\ -2 & 3 & 3 & 3 \end{pmatrix}.$$

EXERCISES (§7.6)

1. Show that the system

$$\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

has solutions only if $\mathbf{b} = (b_1, b_2)^T$ lies on the line $-2b_1 + b_2 = 0$. Is the coefficient matrix singular or nonsingular? Explain.

2. Show that the system

$$\begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

has solutions for all values of b_1 and b_2. Is the coefficient matrix singular or nonsingular? Explain.

3. Show that the system

$$\begin{pmatrix} 3 & 3 & -3 \\ -1 & -1 & 1 \\ 3 & 5 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

has solutions only if $\mathbf{b} = (b_1, b_2, b_3)^T$ lies on the plane $b_1 + 3b_2 = 0$. Is the coefficient matrix singular or nonsingular? Explain.

Using hand calculations only (no computers or calculators), use the technique demonstrated in Examples 6.4 and 6.5 (predicated upon Proposition 6.2) to determine if matrices given in Exercises 4–11 are singular or nonsingular.

4. $\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$

5. $\begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$

6. $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 3 \\ -2 & 3 & 1 \end{pmatrix}$

7. $\begin{pmatrix} 1 & 0 & -1 \\ -2 & 3 & 3 \\ -2 & 3 & 1 \end{pmatrix}$

8. $\begin{pmatrix} 2 & 0 & 2 \\ -1 & 1 & 3 \\ 1 & -4 & 3 \end{pmatrix}$

9. $\begin{pmatrix} 2 & 1 & -1 \\ -1 & -3 & -2 \\ -3 & -2 & 1 \end{pmatrix}$

10. $\begin{pmatrix} -2 & 1 & 0 \\ -2 & 4 & 2 \\ 2 & -2 & 1 \end{pmatrix}$

11. $\begin{pmatrix} -1 & -1 & 1 \\ 0 & -2 & 4 \\ 3 & 0 & 3 \end{pmatrix}$

In Exercises 12–19, find all solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$ for the given coefficient matrix. Does the system have solutions other than the zero vector? Use Proposition 6.6 to determine whether the matrix A is singular or nonsingular.

12. $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$

13. $A = \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}$

14. $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

15. $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

16. $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

17. $A = \begin{pmatrix} 0 & -1 & -2 \\ -5 & 2 & -1 \\ -4 & 2 & 0 \end{pmatrix}$

18. $A = \begin{pmatrix} -2 & -2 & -2 & 0 \\ -1 & -4 & 2 & -3 \\ 3 & 1 & 5 & -2 \\ 3 & 3 & 3 & 0 \end{pmatrix}$

19. $A = \begin{pmatrix} 0 & 1 & 3 & 0 \\ -2 & 1 & -3 & 1 \\ -1 & -2 & 3 & 0 \\ 2 & -1 & -1 & 2 \end{pmatrix}$

In Exercises 20–27, which of the matrices are singular? If a matrix is nonsingular, find its inverse.

20. $A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$

21. $A = \begin{pmatrix} 0 & -4 \\ -1 & 2 \end{pmatrix}$

22. $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

23. $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

24. $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$

25. $A = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

26. $A = \begin{pmatrix} 0 & -3 & -1 & 2 \\ -3 & 0 & 0 & 0 \\ 2 & 1 & -2 & -2 \\ -3 & -1 & 3 & 4 \end{pmatrix}$

27. $A = \begin{pmatrix} 3 & -1 & -3 & -1 \\ 0 & -3 & -4 & 1 \\ -2 & 1 & 2 & 1 \\ 1 & 1 & 3 & -3 \end{pmatrix}$

In Exercises 28–33, without actually solving, which systems have unique solutions? Explain.

28. $\begin{aligned} x_1 + 2x_2 &= 4 \\ x_1 - x_2 &= 6 \end{aligned}$

29. $\begin{aligned} x_1 + 2x_2 &= 4 \\ 2x_1 + 4x_2 &= 8 \end{aligned}$

30. $x_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \end{pmatrix}$

31. $x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

32. $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & -1 \\ 1 & -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

33. $\begin{pmatrix} 1 & 0 & 3 \\ -1 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

34. For which values of x is the matrix

$$\begin{pmatrix} 1 & 3 & -2 \\ 2 & 8 & x \\ 0 & 8 & 5 \end{pmatrix}$$

invertible?

35. List as many properties as you can of an invertible matrix.

36. List as many properties as you can of a nonsingular matrix.

7.7 Determinants

There is a question that is still unanswered. Given a matrix A, is there an easy way to tell if its nullspace is nontrivial? For a square matrix, we found a partial answer in Section 7.6. According to Proposition 6.6, a square matrix has a nontrivial nullspace if and only if it is singular. However, we do not as yet have an easy way to tell if a matrix is singular or nonsingular. Proposition 6.2 tells us that A is nonsingular if and only if when it is transformed into row echelon form, all of the diagonal entries are nonzero, but this is not an adequate answer. A reasonable answer is provided by the determinant.

Let's look at the 2×2 case

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If we assume that $a \neq 0$ and put A into row echelon form with a row operation, we get

$$\begin{pmatrix} a & b \\ 0 & d - bc/a \end{pmatrix}.$$

The diagonal entries will all be nonzero if and only if their product $a \cdot (d - bc/a) = ad - bc \neq 0$. You will recognize that $ad - bc$ is the determinant of A. Thus A is nonsingular if and only if $\det(A) \neq 0$.

Carrying through this calculation for the 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

is tedious, but we again end up with the result that A is nonsingular if and only if $\det(A) \neq 0$. However, now the determinant has the more complicated form

$$\begin{aligned} \det(A) = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} + a_{13}a_{21}a_{32}. \end{aligned} \tag{7.1}$$

(§8.1)

8. Show that the functions $x(t) = (1+t)e^{-t}$ and $y(t) = -te^{-t}$ are solutions of the system

$$x' = y,$$
$$y' = -x - 2y,$$

satisfying the initial conditions $x(0) = 1$ and $y(0) = 0$.

9. Show that the functions $x(t) = e^{-t}(-\cos t - \sin t)$ and $v(t) = 2e^{-t}\sin t$ are solutions of the system

$$x' = v,$$
$$v' = -2x - 2v,$$

satisfying the initial conditions $x(0) = -1$ and $v(0) = 0$.

10. Show that the functions $x(t) = e^{t}$ and $y(t) = e^{-t}$ are solutions of the system

$$x' = x^2 y,$$
$$y' = -xy^2,$$

satisfying the initial conditions $x(0) = 1$ and $y(0) = 1$.

Write each initial value problem in Exercises 11–16 as a system of first-order equations using vector notation.

11. $y'' + 2y' + 4y = 3\cos 2t, \quad y(0) = 1, \quad y'(0) = 0$

12. $mx'' + \mu x' + kx = F_0 \cos \omega t, \quad x(0) = x_0, \quad x'(0) = v_0$

13. $x'' + \delta x' - x + x^3 = \gamma \cos \omega t, \quad x(0) = x_0, \quad x'(0) = v_0$

14. $x'' + \mu(x^2 - 1)x' + x = 0, \quad x(0) = x_0, \quad x'(0) = v_0$

15. $\omega''' = \omega, \quad \omega(0) = \omega_0, \quad \omega'(0) = \alpha_0, \quad \omega''(0) = \gamma_0$

16. $y''' + y'y'' = \sin \omega t, \quad y(0) = \alpha, \quad y'(0) = \beta, \quad y''(0) = \gamma$

Which of the systems in Exercises 17–22 are autonomous? Assume the independent variable is t in each exercise.

17. $u' = v$ and $v' = -3u - 2v + 5\cos t$

18. $u' = u(u^2 + v^2)$ and $v' = -v(u^2 + v^2)$

19. $u' = v\cos(u)$ and $v' = tv$

20. $u' = v\cos(u)$ and $v' = u^2 e^v$

21. $u' = v + \cos(u)$, $v' = v - tw$, and $w' = 5u - 9v + 8w$

22. $u' = w + \cos(u) - 2v$, $v' = u^2 e^v$, and $w' = u + v + w$

23. Write the system in Exercise 17 in vector form.

24. Write the system in Exercise 18 in vector form.

25. Write the system in Exercise 19 in vector form.

26. Write the system in Exercise 20 in vector form.

27. Write the system in Exercise 21 in vector form.

28. Write the system in Exercise 22 in vector form.

29. Use your numerical solver to produce the component solutions for the SIR model pictured in Figure 1.

30. It is sometimes useful to use normalized quantities in the SIR model. To do so, introduce the variables $s = S/N$, $i = I/N$, and $r = R/N$. Each of the new variables represents the fraction of the total population that is in the particular category. Start with the system in (1.1) and derive the system that is satisfied by s, i, and r.

31. A fisherman is located on the opposite bank of a northward flowing river, as shown in Figure 2. The man continually points the nose of his boat toward his destination (the origin at $(0, 0)$), but the current of the river pushes him downstream. Let b be the speed of the boat relative to the water and let a be the speed of the current, both measured in miles per hour. Let $x(t)$ and $y(t)$ denote the x and y position of the boat at time t. Show that the boat obeys the following equations of motion.

$$\frac{dx}{dt} = -\frac{bx}{\sqrt{x^2+y^2}}$$

$$\frac{dy}{dt} = a - \frac{by}{\sqrt{x^2+y^2}}$$

Hint: Break the velocity of the boat into its horizontal and vertical components, find dx/dt and dy/dt in terms of a, b, and θ, and then eliminate θ from the resulting equations.

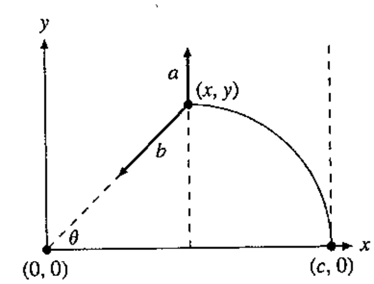

Figure 2. Fighting the current.

32. Use a numerical solver to plot the approximate path of the boat in Exercise 31. Use a number of different values for the parameters a, b, and c. Does the boat always reach its destination (the origin)?

Warning. When the boat reaches its destination, the system of differential equations becomes singular, since the right-hand sides of the equations are discontinuous when x and y are both equal to 0. This can result in your computer exhibiting strange behavior. For example, a variable step algorithm may use smaller and smaller steps, never reaching the end of the computation. Fixed step solvers will reach the end of the computation, but the results will be strange. For this reason, it is best to use final times that underestimate the time it takes to reach the destination. Start with the final time equal to c/b and do the computation a number of times with increasing final times until you are close to the destination. Before even starting this exercise, you should learn how to stop a computation if that becomes necessary.