

HW 8

§ 8.5. 25

§ 9.1. 32

§ 9.2. 3, 16, 45

Extra. $y''' - 4y'' + 3y' = 0$.

$$22. \begin{aligned} y_1(t) &= \begin{pmatrix} e^{-t} \\ 2e^{-t} \\ 2e^{-t} \end{pmatrix}, & y_2(t) &= \begin{pmatrix} e^t \\ e^t \\ 0 \end{pmatrix}, \\ y_3(t) &= \begin{pmatrix} 0 \\ 0 \\ e^t \end{pmatrix} \end{aligned}$$

(5.20)

In Exercises 23–26, verify by direct substitution that $y_1(t)$ and $y_2(t)$ are solutions of the given homogeneous equation. Show also that the solutions $y_1(t)$ and $y_2(t)$ are linearly independent. Find the solution of the given homogeneous equation with the initial condition $y(0) = y_0$.

$$23. \begin{aligned} y_1(t) &= \begin{pmatrix} -e^{2t} \\ 2e^{2t} \end{pmatrix}, & y_2(t) &= \begin{pmatrix} -e^{-2t} \\ e^{-2t} \end{pmatrix}, \\ y' &= \begin{pmatrix} -6 & -4 \\ 8 & 6 \end{pmatrix} y, & \text{and } y(0) &= \begin{pmatrix} -5 \\ 8 \end{pmatrix} \end{aligned}$$

$$24. \begin{aligned} y_1(t) &= \begin{pmatrix} e^{-2t} \\ 0 \end{pmatrix}, & y_2(t) &= \begin{pmatrix} e^{-4t} \\ e^{-4t} \end{pmatrix}, \\ y' &= \begin{pmatrix} -2 & -2 \\ 0 & -4 \end{pmatrix} y, & \text{and } y(0) &= \begin{pmatrix} 3 \\ 1 \end{pmatrix} \end{aligned}$$

$$25. \begin{aligned} y_1(t) &= \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}, & y_2(t) &= \begin{pmatrix} e^{2t}(t+2) \\ e^{2t}(t+1) \end{pmatrix}, \\ y' &= \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} y, & \text{and } y(0) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

$$26. \begin{aligned} y_1(t) &= \begin{pmatrix} \frac{1}{2} \cos t - \frac{1}{2} \sin t \\ \cos t \end{pmatrix}, & y_2(t) &= \begin{pmatrix} \frac{1}{2} \sin t + \frac{1}{2} \cos t \\ \sin t \end{pmatrix}, \\ y' &= \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} y, & \text{and } y(0) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}$$

27. Two tanks each hold 3 liters of salt water and are connected by two pipes (see Figure 1). The salt water in each tank is kept well stirred. Pure water flows into tank A at the rate of 5 liters per minute and the salt water mixture exits tank B at the same rate. Salt water flows from tank A to tank B at the rate of 9 liters per minute and it flows from tank B to tank A at the rate of 4 liters per minute. If tank A initially contains 1 kilogram of salt and tank B contains no salt, then set up the system of differential equations that govern the mass of the salt in each tank $t \geq 0$ minutes later.

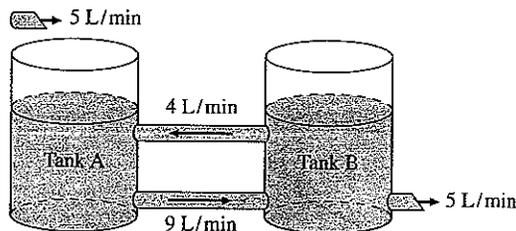


Figure 1. The two tanks in Exercise 27.

28. Suppose that we add an extra tank to the arrangement in Exercise 27, as shown in Figure 2. Further, suppose that rates of flow are defined as follows:

$$\begin{aligned} r_{in} &= 5 \text{ gal/min} & r_{BC} &= 9 \text{ gal/min} \\ r_{AB} &= 9 \text{ gal/min} & r_{CB} &= 4 \text{ gal/min} \\ r_{BA} &= 4 \text{ gal/min} & r_{out} &= 5 \text{ gal/min} \end{aligned}$$

Again, suppose that pure water is pumped into tank A at a rate of r_{in} gal/min. Suppose that each tank initially holds 200 gal of salt solution. Let $x_A(t)$, $x_B(t)$, and $x_C(t)$ represent the amount of salt (in pounds) in tank A, tank B, and tank C, respectively. Set up a system of equations modeling the salt content in each tank over time.

29. Suppose that each tank in Exercise 28 initially contains 40 pounds of salt. Use a numeric solver to plot the salt content of each tank over time.

30. Three tanks containing salt solutions are arranged in a cascade, as shown in Figure 3. Let $x(t)$, $y(t)$, and $z(t)$ represent the amount of salt, in pounds, contained in the upper, middle, and lower tanks, respectively. Pure water enters the upper tank at a rate of 10 gal/min and salt solution leaves the upper tank at the same rate, spilling into the middle tank at 10 gal/min. Salt solution leaves the middle tank at the same rate, spilling into the lower tank at a rate of 10 gal/min. Finally, solution drains from the lower tank at a rate of 10 gal/min. Initially, each tank contains 200 gallons of salt solution. Set up a system of three first-order equations that models the salt content of each tank over time.

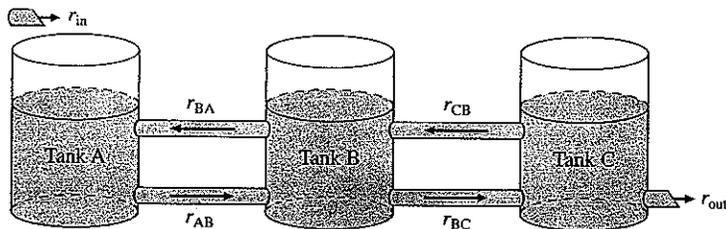


Figure 2. Lots of tanks.

$$26. A = \begin{pmatrix} -1 & 2 & 0 \\ -19 & 14 & 18 \\ 17 & -11 & -17 \end{pmatrix}$$

$$27. A = \begin{pmatrix} -3 & 0 & 2 \\ 6 & 3 & -12 \\ 2 & 2 & -6 \end{pmatrix}$$

Use a computer to find the eigenvalues and eigenvectors for the matrices in Exercises 28–37.

$$28. A = \begin{pmatrix} -6 & 4 & 4 \\ -4 & 2 & 4 \\ -10 & 8 & 4 \end{pmatrix}$$

$$29. A = \begin{pmatrix} -7 & 2 & 10 \\ 0 & 1 & 0 \\ -5 & 2 & 8 \end{pmatrix}$$

$$30. A = \begin{pmatrix} 1 & -2 & 4 \\ 7 & -8 & 10 \\ 2 & -2 & 1 \end{pmatrix}$$

$$31. A = \begin{pmatrix} -11 & -7 & 1 \\ 20 & 13 & -2 \\ -18 & -9 & 2 \end{pmatrix}$$

$$32. A = \begin{pmatrix} -6 & -13 & -11 \\ 4 & 1 & 1 \\ -4 & 2 & 2 \end{pmatrix}$$

$$33. A = \begin{pmatrix} 5 & 0 & 4 \\ -10 & 3 & -8 \\ -14 & 2 & -11 \end{pmatrix}$$

$$34. A = \begin{pmatrix} -2 & -3 & 1 & 1 \\ -4 & -5 & 0 & 4 \\ 7 & 9 & -4 & -9 \\ -7 & -9 & 1 & 6 \end{pmatrix}$$

$$35. A = \begin{pmatrix} -4 & -7 & 0 & 7 \\ 2 & -5 & 2 & 1 \\ -1 & -11 & 1 & 8 \\ 2 & -8 & 2 & 4 \end{pmatrix}$$

$$36. A = \begin{pmatrix} -6 & 5 & -9 & 10 \\ 10 & -7 & 13 & -16 \\ 4 & -4 & 8 & -8 \\ -5 & 3 & -5 & 7 \end{pmatrix}$$

$$37. A = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 0 & 2 & 2 \\ 1 & 3 & 0 & 2 \\ 4 & -8 & 4 & 2 \end{pmatrix}$$

Use a computer to find a fundamental set of solutions for the system $\mathbf{y}' = A\mathbf{y}$, where A is the matrix given in Exercises 38–47.

$$38. A = \begin{pmatrix} -2 & -6 & 12 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}$$

$$39. A = \begin{pmatrix} 20 & -34 & -10 \\ 12 & -21 & -5 \\ -2 & 4 & -2 \end{pmatrix}$$

$$40. A = \begin{pmatrix} 0 & -2 & 2 \\ 12 & -6 & 8 \\ 9 & -1 & 3 \end{pmatrix}$$

$$41. A = \begin{pmatrix} -3 & -10 & 0 \\ 0 & 2 & 0 \\ 2 & 4 & -1 \end{pmatrix}$$

$$42. A = \begin{pmatrix} -5 & 6 & 4 \\ -18 & 16 & 8 \\ 72 & -48 & -13 \end{pmatrix}$$

$$43. A = \begin{pmatrix} 7 & 7 & 4 \\ -6 & -10 & -8 \\ 6 & 7 & 5 \end{pmatrix}$$

$$44. A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 3 & 0 & 2 & -1 \\ -4 & -2 & 5 & -8 \\ -2 & -2 & 4 & -7 \end{pmatrix}$$

$$45. A = \begin{pmatrix} 5 & 7 & -2 & 8 \\ -6 & -8 & 2 & -14 \\ 6 & 6 & -4 & 6 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$46. A = \begin{pmatrix} -16 & 24 & -48 & 7 \\ 10 & -7 & 23 & -5 \\ 10 & -10 & 26 & -5 \\ 6 & 4 & 8 & -5 \end{pmatrix}$$

$$47. A = \begin{pmatrix} 6 & 4 & -8 & 0 \\ 14 & 2 & 22 & -18 \\ 11 & 4 & 5 & -9 \\ 22 & 8 & 2 & -14 \end{pmatrix}$$

48. Use Definition 1.4 to show that if \mathbf{v} and \mathbf{w} are eigenvectors of A associated to the eigenvalue λ , then $a\mathbf{v} + b\mathbf{w}$ is also an eigenvector associated to λ for any scalars a and b .

49. Use a computer to find the eigenvalues and determinant of each of the following matrices:

$$A = \begin{pmatrix} 6 & -8 \\ 4 & -6 \end{pmatrix}, \quad B = \begin{pmatrix} -11 & -16 \\ 8 & 13 \end{pmatrix},$$

$$\text{and } C = \begin{pmatrix} 7 & -21 & -11 \\ 5 & -13 & -5 \\ -5 & 9 & 1 \end{pmatrix}.$$

Describe any relationship you see between the eigenvalues and the determinant.

50. The *trace* of a matrix is the sum of the elements on its main diagonal ($\text{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$). For each matrix in Exercise 49, describe a relationship between the eigenvalues and the trace of the matrix.

In Example 2.30 we found that the vector of currents $\mathbf{I} = (I_1, I_2)^T$, satisfies $\mathbf{I}' = A\mathbf{I}$, where

$$A = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}.$$

We also found that A has a single eigenvalue $\lambda_1 = -2$ and the eigenvector $\mathbf{v}_1 = (1, 1)^T$. The corresponding solution is

$$\mathbf{x}_1(t) = e^{\lambda_1 t} \mathbf{v}_1 = e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

According to Theorem 2.40, we need to find a vector \mathbf{v}_2 which satisfies $(A - \lambda_1 I)\mathbf{v}_2 = \mathbf{v}_1$. The process for doing this is covered in the paragraph containing formula (2.39). We can start with literally any vector \mathbf{w} that is not a multiple of \mathbf{v}_1 . Given this flexibility, we should choose a vector that contains as many zeros as possible to facilitate computation. For example, if we choose a simple vector like $\mathbf{w} = (0, 1)^T$, we proceed as follows. Compute

$$(A - \lambda_1 I)\mathbf{w} = (A + 2I)\mathbf{w} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} = -1\mathbf{v}_1.$$

Hence, we take $\mathbf{v}_2 = (-1)\mathbf{w} = (0, -1)^T$.

Our fundamental set of solutions is

$$\begin{aligned} \mathbf{x}_1(t) &= e^{\lambda_1 t} \mathbf{v}_1 = e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \\ \mathbf{x}_2(t) &= e^{\lambda_1 t} (\mathbf{v}_2 + t\mathbf{v}_1) = e^{-2t} \begin{pmatrix} t \\ t-1 \end{pmatrix}. \end{aligned} \tag{2.44}$$

The general solution is

$$\begin{aligned} \mathbf{I}(t) &= C_1 \mathbf{x}_1(t) + C_2 \mathbf{x}_2(t) \\ &= e^{-2t} \left(C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} t \\ t-1 \end{pmatrix} \right). \end{aligned} \tag{2.45}$$

$$\tag{2.46}$$

At $t = 0$ we have

$$\begin{pmatrix} 5 \\ 1 \end{pmatrix} = \mathbf{I}(0) = \begin{pmatrix} C_1 \\ C_1 - C_2 \end{pmatrix}.$$

Hence, $C_1 = 5$ and $C_2 = 4$. The solution with these initial values is

$$\mathbf{I}(t) = e^{-2t} \begin{pmatrix} 5 + 4t \\ 1 + 4t \end{pmatrix}.$$

The components of the solution are plotted in Figure 6.

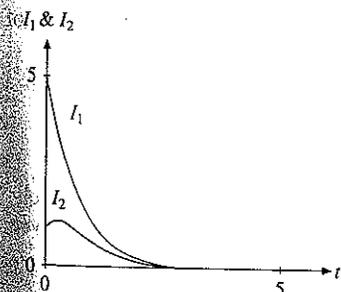


Figure 6. The currents in Example 2.43.

EXERCISES

In Exercises 1–6, the matrix A has real eigenvalues. Find the general solution of the system $\mathbf{y}' = A\mathbf{y}$.

1. $A = \begin{pmatrix} 2 & -6 \\ 0 & -1 \end{pmatrix}$

2. $A = \begin{pmatrix} -1 & 6 \\ -3 & 8 \end{pmatrix}$

3. $A = \begin{pmatrix} -5 & 1 \\ -2 & -2 \end{pmatrix}$

5. $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$

4. $A = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$

6. $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

(89.2)

In Exercises 7–12, find the solution of the initial-value problem for system $y' = Ay$ with the given matrix A and the given initial value.

7. The matrix in Exercise 1 with $y(0) = (0, 1)^T$
8. The matrix in Exercise 2 with $y(0) = (1, -2)^T$
9. The matrix in Exercise 3 with $y(0) = (0, -1)^T$
10. The matrix in Exercise 4 with $y(0) = (1, 1)^T$
11. The matrix in Exercise 5 with $y(0) = (3, 2)^T$
12. The matrix in Exercise 6 with $y(0) = (1, 5)^T$

In Exercises 13 and 14, a complex vector valued function $z(t)$ is given. Find the real and imaginary parts of $z(t)$.

13. $z(t) = e^{2it} \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$ 14. $z(t) = e^{(1+i)t} \begin{pmatrix} -1+i \\ 2 \end{pmatrix}$

15. The system

$$y' = \begin{pmatrix} 3 & 3 \\ -6 & -3 \end{pmatrix} y \quad (2.47)$$

has complex solution

$$z(t) = e^{3it} \begin{pmatrix} -1-i \\ 2 \end{pmatrix}.$$

Verify, by direct substitution, that the real and imaginary parts of this solution are solutions of system (2.47). Then use Proposition 5.2 in Section 8.5 to verify that they are linearly independent solutions.

In Exercises 16–21, the matrix A has complex eigenvalues. Find a fundamental set of real solutions of the system $y' = Ay$.

16. $A = \begin{pmatrix} -4 & -8 \\ 4 & 4 \end{pmatrix}$
17. $A = \begin{pmatrix} -1 & -2 \\ 4 & 3 \end{pmatrix}$
18. $A = \begin{pmatrix} -1 & 1 \\ -5 & -5 \end{pmatrix}$
19. $A = \begin{pmatrix} 0 & 4 \\ -2 & -4 \end{pmatrix}$
20. $A = \begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix}$
21. $A = \begin{pmatrix} 3 & -6 \\ 3 & 5 \end{pmatrix}$

In Exercises 22–27, find the solution of the initial value problem for system $y' = Ay$ with the given matrix A and the given initial value.

22. The matrix in Exercise 16 with $y(0) = (0, 2)^T$
23. The matrix in Exercise 17 with $y(0) = (0, 1)^T$
24. The matrix in Exercise 18 with $y(0) = (1, -5)^T$
25. The matrix in Exercise 19 with $y(0) = (-1, 2)^T$
26. The matrix in Exercise 20 with $y(0) = (3, 2)^T$
27. The matrix in Exercise 21 with $y(0) = (1, 3)^T$
28. Suppose that A is a real 2×2 matrix with one eigenvalue λ of multiplicity two. Show that the solution to the initial value problem $y' = Ay$ with $y(0) = v$ is given by

$$y(t) = e^{\lambda t} [v + t(A - \lambda I)v].$$

Hint: Verify the result by direct substitution. Remember that $(A - \lambda I)^2 = 0I$, so $A(A - \lambda I) = \lambda(A - \lambda I)$.

In Exercises 29–34 the matrix A has one real eigenvalue of multiplicity two. Find the general solution of the system $y' = Ay$.

29. $A = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$
30. $A = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix}$
31. $A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$
32. $A = \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}$
33. $A = \begin{pmatrix} -2 & 1 \\ -9 & 4 \end{pmatrix}$
34. $A = \begin{pmatrix} 5 & 1 \\ -4 & 1 \end{pmatrix}$

In Exercises 35–40, find the solution of the initial value problem for system $y' = Ay$ with the given matrix A and the given initial value.

35. The matrix in Exercise 29 with $y(0) = (3, -2)^T$
36. The matrix in Exercise 30 with $y(0) = (0, -3)^T$
37. The matrix in Exercise 31 with $y(0) = (2, -1)^T$
38. The matrix in Exercise 32 with $y(0) = (1, 1)^T$
39. The matrix in Exercise 33 with $y(0) = (5, 3)^T$
40. The matrix in Exercise 34 with $y(0) = (0, 2)^T$

In Exercises 41–48, find the general solution of the system $y' = Ay$ for the given matrix A .

41. $A = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix}$
42. $A = \begin{pmatrix} -8 & -10 \\ 5 & 7 \end{pmatrix}$
43. $A = \begin{pmatrix} 5 & 12 \\ -4 & -9 \end{pmatrix}$
44. $A = \begin{pmatrix} -6 & 1 \\ 0 & -6 \end{pmatrix}$
45. $A = \begin{pmatrix} -4 & -5 \\ 2 & 2 \end{pmatrix}$
46. $A = \begin{pmatrix} -6 & 4 \\ -8 & 2 \end{pmatrix}$
47. $A = \begin{pmatrix} -10 & 4 \\ -12 & 4 \end{pmatrix}$
48. $A = \begin{pmatrix} -1 & 5 \\ -5 & -1 \end{pmatrix}$

In Exercises 49–56, find the solution of the initial value problem for system $y' = Ay$ with the given matrix A and the given initial value.

49. The matrix in Exercise 41 with $y(0) = (3, 1)^T$
50. The matrix in Exercise 42 with $y(0) = (3, 1)^T$
51. The matrix in Exercise 43 with $y(0) = (1, 0)^T$
52. The matrix in Exercise 44 with $y(0) = (1, 0)^T$
53. The matrix in Exercise 45 with $y(0) = (-3, 2)^T$
54. The matrix in Exercise 46 with $y(0) = (4, 0)^T$
55. The matrix in Exercise 47 with $y(0) = (2, 1)^T$
56. The matrix in Exercise 48 with $y(0) = (5, 5)^T$
57. The Cayley-Hamilton theorem is one of the most important results in linear algebra. The proof in general is quite difficult, but for the case of a 2 matrix with a single eigenvalue λ of multiplicity 2, the proof is not so bad. We need to show that $(A - \lambda I)^2 = 0I$.

(a) Show that it is enough to show that $(A - \lambda I)^2 v = 0$ for every vector $v \in \mathbb{R}^2$.

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