HW 9:

§9.2. 30, 52

§9.4. 3, 4, 5, 6.

↑ (for 3, 4, 5, 6) only need sketch the phase portrait by hand.

(For Nodal Sink & Sources, draw precisely!)

(For Spiral Sink & Sources, draw rotation direction precisely!)

In Exercises 7–12, find the solution of the initial-value problem for system $\mathbf{y}' = A\mathbf{y}$ with the given matrix A and the given initial value.

7. The matrix in Exercise 1 with $\mathbf{y}(0) = (0, 1)^T$
8. The matrix in Exercise 2 with $\mathbf{y}(0) = (1, -2)^T$
9. The matrix in Exercise 3 with $\mathbf{y}(0) = (0, -1)^T$
10. The matrix in Exercise 4 with $\mathbf{y}(0) = (1, 1)^T$
11. The matrix in Exercise 5 with $\mathbf{y}(0) = (3, 2)^T$
12. The matrix in Exercise 6 with $\mathbf{y}(0) = (1, 5)^T$

In Exercises 13 and 14, a complex vector valued function $\mathbf{z}(t)$ is given. Find the real and imaginary parts of $\mathbf{z}(t)$.

13. $\mathbf{z}(t) = e^{2it}\begin{pmatrix} 1 \\ 1+i \end{pmatrix}$ 14. $\mathbf{z}(t) = e^{(1+i)t}\begin{pmatrix} -1+i \\ 2 \end{pmatrix}$

15. The system

$$\mathbf{y}' = \begin{pmatrix} 3 & 3 \\ -6 & -3 \end{pmatrix}\mathbf{y} \tag{2.47}$$

has complex solution

$$\mathbf{z}(t) = e^{3it}\begin{pmatrix} -1-i \\ 2 \end{pmatrix}.$$

Verify, by direct substitution, that the real and imaginary parts of this solution are solutions of system (2.47). Then use Proposition 5.2 in Section 8.5 to verify that they are linearly independent solutions.

In Exercises 16–21, the matrix A has complex eigenvalues. Find a fundamental set of real solutions of the system $\mathbf{y}' = A\mathbf{y}$.

16. $A = \begin{pmatrix} -4 & -8 \\ 4 & 4 \end{pmatrix}$ 17. $A = \begin{pmatrix} -1 & -2 \\ 4 & 3 \end{pmatrix}$

18. $A = \begin{pmatrix} -1 & 1 \\ -5 & -5 \end{pmatrix}$ 19. $A = \begin{pmatrix} 0 & 4 \\ -2 & -4 \end{pmatrix}$

20. $A = \begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix}$ 21. $A = \begin{pmatrix} 3 & -6 \\ 3 & 5 \end{pmatrix}$

In Exercises 22–27, find the solution of the initial value problem for system $\mathbf{y}' = A\mathbf{y}$ with the given matrix A and the given initial value.

22. The matrix in Exercise 16 with $\mathbf{y}(0) = (0, 2)^T$
23. The matrix in Exercise 17 with $\mathbf{y}(0) = (0, 1)^T$
24. The matrix in Exercise 18 with $\mathbf{y}(0) = (1, -5)^T$
25. The matrix in Exercise 19 with $\mathbf{y}(0) = (-1, 2)^T$
26. The matrix in Exercise 20 with $\mathbf{y}(0) = (3, 2)^T$
27. The matrix in Exercise 21 with $\mathbf{y}(0) = (1, 3)^T$
28. Suppose that A is a real 2×2 matrix with one eigenvalue λ of multiplicity two. Show that the solution to the initial value problem $\mathbf{y}' = A\mathbf{y}$ with $\mathbf{y}(0) = \mathbf{v}$ is given by

$$\mathbf{y}(t) = e^{\lambda t}[\mathbf{v} + t(A - \lambda I)\mathbf{v}].$$

Hint: Verify the result by direct substitution. Remember that $(A - \lambda I)^2 = 0I$, so $A(A - \lambda I) = \lambda(A - \lambda I)$.

In Exercises 29–34 the matrix A has one real eigenvalue of multiplicity two. Find the general solution of the system $\mathbf{y}' = A\mathbf{y}$.

29. $A = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ 30. $A = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix}$

31. $A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$ 32. $A = \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}$

33. $A = \begin{pmatrix} -2 & 1 \\ -9 & 4 \end{pmatrix}$ 34. $A = \begin{pmatrix} 5 & 1 \\ -4 & 1 \end{pmatrix}$

In Exercises 35–40, find the solution of the initial value problem for system $\mathbf{y}' = A\mathbf{y}$ with the given matrix A and the given initial value.

35. The matrix in Exercise 29 with $\mathbf{y}(0) = (3, -2)^T$
36. The matrix in Exercise 30 with $\mathbf{y}(0) = (0, -3)^T$
37. The matrix in Exercise 31 with $\mathbf{y}(0) = (2, -1)^T$
38. The matrix in Exercise 32 with $\mathbf{y}(0) = (1, 1)^T$
39. The matrix in Exercise 33 with $\mathbf{y}(0) = (5, 3)^T$
40. The matrix in Exercise 34 with $\mathbf{y}(0) = (0, 2)^T$

In Exercises 41–48, find the general solution of the system $\mathbf{y}' = A\mathbf{y}$ for the given matrix A.

41. $A = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix}$ 42. $A = \begin{pmatrix} -8 & -10 \\ 5 & 7 \end{pmatrix}$

43. $A = \begin{pmatrix} 5 & 12 \\ -4 & -9 \end{pmatrix}$ 44. $A = \begin{pmatrix} -6 & 1 \\ 0 & -6 \end{pmatrix}$

45. $A = \begin{pmatrix} -4 & -5 \\ 2 & 2 \end{pmatrix}$ 46. $A = \begin{pmatrix} -6 & 4 \\ -8 & 2 \end{pmatrix}$

47. $A = \begin{pmatrix} -10 & 4 \\ -12 & 4 \end{pmatrix}$ 48. $A = \begin{pmatrix} -1 & 5 \\ -5 & -1 \end{pmatrix}$

In Exercises 49–56, find the solution of the initial value problem for system $\mathbf{y}' = A\mathbf{y}$ with the given matrix A and the given initial value.

49. The matrix in Exercise 41 with $\mathbf{y}(0) = (3, 1)^T$
50. The matrix in Exercise 42 with $\mathbf{y}(0) = (3, 1)^T$
51. The matrix in Exercise 43 with $\mathbf{y}(0) = (1, 0)^T$
52. The matrix in Exercise 44 with $\mathbf{y}(0) = (1, 0)^T$
53. The matrix in Exercise 45 with $\mathbf{y}(0) = (-3, 2)^T$
54. The matrix in Exercise 46 with $\mathbf{y}(0) = (4, 0)^T$
55. The matrix in Exercise 47 with $\mathbf{y}(0) = (2, 1)^T$
56. The matrix in Exercise 48 with $\mathbf{y}(0) = (5, 5)^T$
57. The Cayley-Hamilton theorem is one of the most important results in linear algebra. The proof in general is quite difficult, but for the case of a 2 matrix with a single eigenvalue λ of multiplicity 2, the proof is not so bad. We need to show that $(A - \lambda I)^2 = 0I$.

    (a) Show that it is enough to show that $(A - \lambda I)^2\mathbf{v} = \mathbf{0}$ for every vector $\mathbf{v} \in \mathbf{R}^2$.

(b) Choose
$\lambda I)^2\mathbf{v} =$
(c) Suppos
and $\mathbf{v}_1$
(d) Set $\mathbf{w} =$
and b s
(e) Show t
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(f) Conclu

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EXERCISES (§9.4)

In Exercises 1–12, classify the equilibrium point of the system $\mathbf{y}' = A\mathbf{y}$ based on the position of (T, D) in the trace-determinant plane. Sketch the phase portrait by hand. Verify your result by creating a phase portrait with your numerical solver.

1. $A = \begin{pmatrix} 8 & 20 \\ -4 & -8 \end{pmatrix}$

2. $A = \begin{pmatrix} -16 & 9 \\ -18 & 11 \end{pmatrix}$

3. $A = \begin{pmatrix} 2 & -4 \\ 8 & -6 \end{pmatrix}$

4. $A = \begin{pmatrix} 8 & 3 \\ -6 & -1 \end{pmatrix}$

5. $A = \begin{pmatrix} -11 & -5 \\ 10 & 4 \end{pmatrix}$

6. $A = \begin{pmatrix} 6 & -5 \\ 10 & -4 \end{pmatrix}$

7. $A = \begin{pmatrix} -7 & 10 \\ -5 & 8 \end{pmatrix}$

8. $A = \begin{pmatrix} 4 & 3 \\ -15 & -8 \end{pmatrix}$

9. $A = \begin{pmatrix} 3 & 2 \\ -4 & -1 \end{pmatrix}$

10. $A = \begin{pmatrix} -5 & 2 \\ -6 & 2 \end{pmatrix}$

11. $A = \begin{pmatrix} -4 & 10 \\ -2 & 4 \end{pmatrix}$

12. $A = \begin{pmatrix} -2 & -6 \\ 4 & 8 \end{pmatrix}$

Degenerate nodes. Degenerate nodal sinks are equilibrium points characterized by the fact that all solutions tend toward the equilibrium point as $t \to \infty$, all tangent to the same line. The solutions of degenerate nodal sources tend toward the equilibrium point as $t \to -\infty$, all tangent to the same line. Exercises 13–19 discuss degenerate nodes.

13. The system

$$\mathbf{y}' = \begin{pmatrix} 1 & 4 \\ -1 & -3 \end{pmatrix} \mathbf{y}$$

has a repeated eigenvalue, $\lambda = -1$, but only one eigenvector, $\mathbf{v}_1 = (2, -1)^T$.

   (a) Explain why this system lies on the boundary separating nodal sinks from spiral sinks in the trace-determinant plane.

   (b) The general solution (see (2.42) in Section 9.2) can be written

$$\mathbf{y}(t) = e^{-t}\left((C_1 + C_2 t)\begin{pmatrix} 2 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}\right).$$

   Predict the behavior of the solution in the phase plane as $t \to \infty$ and as $t \to -\infty$.

   (c) Use your numerical solver to sketch the half-line solutions. Then sketch exactly one solution in each region separated by the half-line solutions. Explain how the behavior you see agrees with your findings in part (b).

In Exercises 14–16, let A be a real 2×2 matrix with one eigenvalue λ, and suppose that the eigenspace of λ has dimension 1. The matrix in Exercise 13 is an example. According to Theorem 2.40 in Section 9.2, a fundamental set of solutions for the system $\mathbf{y}' = A\mathbf{y}$ is given by

$$\mathbf{y}_1(t) = e^{\lambda t}\mathbf{v}_1 \quad \text{and}$$
$$\mathbf{y}_2(t) = e^{\lambda t}[\mathbf{v}_2 + t\mathbf{v}_1],$$

where $\mathbf{v}_1$ is a nonzero eigenvector and $\mathbf{v}_2$ satifies $(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$. By (2.42), the general solution can be written as

$$\mathbf{y}(t) = e^{\lambda t}[(C_1 + C_2 t)\mathbf{v}_1 + C_2\mathbf{v}_2].$$

14. Assume that the eigenvalue λ is negative.

   (a) Describe the exponential solutions.

   (b) Describe the behavior of the general solution as $t \to \infty$.

   (c) Describe the behavior of the general solution as $t \to -\infty$.

   (d) Is the equilibrium point at the origin a degenerate nodal sink or source?

15. Redo Exercise 14 under the assumption that the eigenvalue λ is positive.

16. Where do linear degenerate sources and sinks fit on the trace-determinant plane?

In Exercises 17–18, find the general solution of the given system. Write your solution in the form

$$\mathbf{y}(t) = e^{\lambda t}[(C_1 + C_2 t)\mathbf{v}_1 + C_2\mathbf{v}_2],$$

where $\mathbf{v}_1$ is an eigenvector and $\mathbf{v}_2$ satisfies $(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$. Without the use of a computer or a calculator, sketch the half-line solutions. Sketch exactly one solution in each region separated by the half-line solutions. Use a numerical solver to verify your result when finished. *Hint:* The solutions in this case want desperately to spiral but are prevented from doing so by the half-line solutions (solutions cannot cross). However, the suggestions regarding clockwise or counterclockwise rotation in the subsection on spiral sources apply nicely in this situation.

17. $\mathbf{y}' = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix}\mathbf{y}$

18. $\mathbf{y}' = \begin{pmatrix} -4 & -4 \\ 1 & 0 \end{pmatrix}\mathbf{y}$

19. Consider the system

$$x' = x + ay,$$
$$y' = x + y.$$

   (a) Show that the equilibrium point is a nodal source for all $0 < a < 1$.

   (b) In the case $0 < a < 1$, what are the equations of the half-line solutions? Explain what happens to the half-line solutions as $a \to 0$.

   (c) What happens to the system when $a = 0$? When $a < 0$?

20. **Star nodes** A star sink is an equilibrium point that has the property that there are solution curves tending to it as $t \to \infty$ tangent to any direction. A star source is defined