Laplace Transform.

Motivation: Consider Differential Equation $y'' + y' = f(t)$.

It's hard to compute $y(t)$ to get solution.

Now if we can find a way which turn differential operation into multiplication, then things become totally different.

Now see how cool Laplace Transform is!

$\boxed{\text{Defn.}}$ Suppose $f(t)$ is a fcn of t defined for $0 < t < \infty$.

The Laplace transform of f is the fcn

$$\mathcal{L}\{f\}(t) = F(s) = \int_0^\infty f(t) e^{-ts} \, dt, \text{ for } s > 0.$$

Let's see the essential property of $\mathcal{L}(f)$.

First $\quad \mathcal{L}(f')(s) = \int_0^\infty f'(t) e^{-ts}\, dt$

use integration by parts
应用分部积分

$$= f(t)e^{-ts}\Big|_0^\infty - \int_0^\infty f(t)\cdot(-s)e^{-ts}\,dt$$

$$= f(t)e^{-ts}\Big|_0^\infty + s\underbrace{\int_0^\infty f(t)e^{-ts}\,dt}_{\mathcal{L}(f)}$$

$$= s\mathcal{L}(f)(s) + f(t)e^{-ts}\Big|_0^\infty = s\mathcal{L}(f)(s) - f(0)$$

$$(e^{-ts}|_\infty = 0).$$

Second $\quad \mathcal{L}(f'')(s) = \int_0^\infty f''(t) e^{-ts}\, dt$

$$= f'(t)e^{-ts}\Big|_0^\infty - \int_0^\infty f'(t)\cdot(-s)e^{-ts}\,dt$$

$$= f'(t)e^{-ts}\Big|_0^\infty + s\underbrace{\int_0^\infty f'(t)e^{-ts}\,dt}_{\mathcal{L}(f')}$$

$$= s\mathcal{L}(f')(s) + f'(t)e^{-ts}\Big|_0^\infty = s\mathcal{L}(f')(s) - f'(0)$$

For the differential equation $y'' + y' = f(t)$.

$\boxed{\text{Do Laplace Tranform on both sides}}$

$$\mathcal{L}(y'')(s) + \mathcal{L}(y')(s) = \mathcal{L}(f)(s)$$

$$\Rightarrow (s\mathcal{L}(y')(s) - y'(0)) + \mathcal{L}(y')(s) = \mathcal{L}(f)(s)$$

$$\Rightarrow (s+1)\mathcal{L}(y')(s) - y'(0) = \mathcal{L}(f)(s)$$

$$\Rightarrow (s+1)(s\mathcal{L}(y)(s) - y(0)) - y'(0) = \mathcal{L}(f)(s).$$

$$\Rightarrow \quad (s+1)s\,\mathcal{L}(y)(s) - (s+1)y(0) - y'(0) = \mathcal{L}(f)(s)$$

$$\Rightarrow \quad (s+1)s\,\mathcal{L}(y)(s) = (s+1)y(0) + y'(0) + \mathcal{L}(f)(s)$$

$$\Rightarrow \quad \mathcal{L}(y)(s) = \frac{1}{(s+1)s}\left[(s+1)y(0) + y'(0) + \mathcal{L}(f)(s)\right].$$

Conclusion: If we know $\mathcal{L}(f)(s)$ for given f, $y(0)$, $y'(0)$, then

we know $\mathcal{L}(y)(s)$.

In addition, if we know what's y for given

$\mathcal{L}(y)(s)$, then we've done looking for solution y.

The following notes more concern about the calculation of

$\mathcal{L}(f)(t)$ for given f,

like $f(t) = \cos t$, $\sin t$, t^n, ...

Ex. $\mathcal{L}(f)(s)$, for $f = t$.

$$\mathcal{L}(f)(s) = \mathcal{L}(t)(s) = \int_0^\infty t e^{-ts}\,dt$$

integration by parts

$$= -\frac{1}{s}e^{-ts}\cdot t\Big|_0^\infty + \int_0^\infty \frac{1}{s}e^{-ts}\,dt$$

$$= -\frac{t}{s}e^{-ts}\Big|_{t\,0}^{\infty} - \frac{1}{s^2}e^{-ts}\Big|_{t\,0}^{\infty}$$

$$= 0 + \frac{1}{s^2} = \frac{1}{s^2}.$$

Ex. $\mathcal{L}(f)(s)$, for $f = t^n$.

Similarly, we can get $\mathcal{L}(t^n)(s) = \frac{n!}{s^{n+1}}$.

Ex. $\mathcal{L}(f)(s)$, for $f = e^{at}$.

$$\mathcal{L}(f)(s) = \mathcal{L}(e^{at})(s) = \int_0^\infty e^{(a-s)t}\,dt$$

integration by parts

$$= \frac{1}{a-s}e^{(a-s)t}\Big|_{t\,0}^{\infty}$$

If $a \geq s$, then $\mathcal{L}(f)(s)$ is undefined.

If $a < s$, then $\mathcal{L}(f)(s) = \frac{1}{a-s}\cdot(-1) = \frac{1}{s-a}$.

so $\mathcal{L}(f)(s) = F(s) = \frac{1}{s-a}$ for $s > a$.

Remark: The method of integration by parts is often useful when computing Laplace Transforms.

We remind you that it says

$$\int u\,dv = uv - \int v\,du.$$

where u, v are fcns.

Ex. $\mathcal{L}(f)(s)$, for $f(t) = \sin at$.

$$\mathcal{L}(f)(s) = \int \sin at\, e^{-st}\,dt$$

$$= -\frac{1}{s} e^{-st} \sin at \Big|_0^\infty + \int \frac{1}{s} e^{-st} a\cos at\,dt$$

$$= \frac{a}{s} \int e^{-st} \cos at\,dt. \quad (*)$$

We still don't know $\int e^{-st}\cos at\,dt$, let's calculate it again.

$$\int e^{-st}\cos at\,dt = -\frac{1}{s} e^{-st}\cos at \Big|_0^\infty - \int \frac{1}{s} e^{-st} \cdot a\sin at\,dt$$

$$= \frac{1}{s} - \frac{a}{s}\int e^{-st}\sin at\,dt$$

We also know from $(*)$, $\int e^{-st}\cos at\,dt = \frac{s}{a}\int \sin at\, e^{-st}\,dt$

so $\left(\frac{s}{a} + \frac{a}{s}\right)\int e^{-st}\sin at\,dt = \frac{1}{s}$

$$\Rightarrow \mathcal{L}(f)(s) = \int e^{-st}\sin at\,dt = \frac{a}{s^2+a^2}.$$

Same method, we can compute

$$\mathcal{L}(\cos at)(s) = \frac{s}{s^2+a^2}.$$

<u>The Existence of Laplace tranform</u>. (of discontinuous fcns.)

Ex. $\mathcal{L}(f)(s)$, where $f(t) = \begin{cases} 1, & \text{for } 0 \le t < 1 \\ 0, & \text{for } t \ge 1. \end{cases}$

$$\mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-ts}\,dt = \int_0^1 e^{-ts}\,dt$$

$$= -\frac{1}{s}e^{-ts}\Big|_{t\,0}^{1} = -\frac{1}{s}(e^{-s}-1) = \frac{1}{s}(1-e^{-s}).$$

Q: Does any function have Laplace Tranform?

A: By the example above, the fcn is not necessarily continuous.

But we still have some restrictions on fcns which have Laplace Transform.

That is "of exponential order".

Defn. A fcn $f(t)$ is of exponential order if there are constants C and a s.t $|f(t)| \le Ce^{at}$ for all $t>0$.

Ex. $f(t) = e^{t^2}$. is not of exponential order.

Ex. $f(t) = e^{\sqrt{t}}$. is of exponential order.

Thm. Suppose f is a piecewise continuous fcn defined on $[0,\infty)$, which is of exponential order. Then the Laplace Transform $\mathcal{L}(f)(s)$ exists for large values of s. Sepecially, if $|f(t)| \le Ce^{at}$, then $\mathcal{L}(f)(s)$ exists at least for $s>a$.

Homework:

1. $\mathcal{L}(f)(s)$, for $f(t) = \cos 3t$.
2. $\mathcal{L}(f)(s)$, for $f(t) = te^{-3t}$.
3. $\mathcal{L}(f)(s)$, for $f(t) = e^{2t}\cos 3t$.

Properties of Laplace Transform.

Prop. Suppose f and g are piecewise continuous fcns of exponential order, and α, β are constants. Then

$$\mathcal{L}\{\alpha f(t)+\beta g(t)\}(s) = \alpha \mathcal{L}\{f(t)\}(s) + \beta \mathcal{L}\{g(t)\}(s)$$

Pf:

$$\mathcal{L}\{\alpha f(t)+\beta g(t)\}(s) = \int_0^\infty (\alpha f(t)+\beta g(t))\, e^{-st}\, dt$$

$$= \alpha \int_0^\infty f(t) e^{-st}\, dt + \beta \int_0^\infty f(t) e^{-st}\, dt$$

$$= \alpha \mathcal{L}\{f(t)\}(s) + \beta \mathcal{L}\{g(t)\}(s).$$

Ex. $\mathcal{L}(f)(s)$, for $f(t) = 3\sin 2t - 4t + 5e^{3t}$.

$$\mathcal{L}(f)(s) \overset{\text{by Proposition}}{=} \mathcal{L}(3\sin 2t)(s) + \mathcal{L}(-4t)(s) + \mathcal{L}(5e^{3t})(s)$$

$$= 3\,\mathcal{L}(\sin 2t)(s) - 4\,\mathcal{L}(t)(s) + 5\,\mathcal{L}(e^{3t})(s)$$

$$= 3\cdot\frac{2}{4+s^2} - 4\cdot\frac{1}{s^2} + 5\cdot\frac{1}{s-3}.$$

$$= \frac{6}{4+s^2} - \frac{6}{s^2} + \frac{5}{s-3}.$$

From our motivation, we notice it's also important to know how to do the calculation of The Inverse Laplace Transform.

Defn. If f is a continuous fcn of exponential order and $\mathcal{L}(f)(s) = F(s)$, then we call f the inverse Laplace transform of F, and write $f = \mathcal{L}^{-1}(F)$.

We also have the linearity of inverse Laplace Transform.

Thm. $\mathcal{L}^{-1}(aF + bG) = a\mathcal{L}^{-1}(F) + b\mathcal{L}^{-1}(G)$, where a, b are arbitrary contants.

Table: A small table of Laplace transforms.

$f(t)$	$\mathcal{L}(f)(s) = F(s)$
1	$\frac{1}{s}$, $s>0$
t^n	$\frac{n!}{s^{n+1}}$, $s>0$
$\sin at$	$\frac{a}{s^2+a^2}$, $s>0$
$\cos at$	$\frac{s}{s^2+a^2}$, $s>0$
e^{at}	$\frac{1}{s-a}$, $s>a$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$, $s>a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$, $s>a$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$, $s>a$

Ex. Compute $\mathcal{L}^{-1}(F(s))(t)$, where $F(s) = \frac{1}{s-2} - \frac{16}{s^2+4}$.

so $\mathcal{L}^{-1}(F(s))(t)$

$$= \mathcal{L}^{-1}\left(\frac{1}{s-2}\right)(t) - \mathcal{L}^{-1}\left(\frac{16}{s^2+4}\right)(t)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s-2}\right)(t) - 8\,\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right)(t)$$

From the table $= e^{2t} - 8\sin 2t$.

Ex. Compute $\mathcal{L}^{-1}(F(s))(t)$, where $F(s) = \frac{1}{s^2+4s+13}$

$\mathcal{L}^{-1}(F(s))(t)$ $\qquad s^2+4s+13 = (s+2)^2+3^2$

$$= \mathcal{L}^{-1}\left(\frac{1}{(s+2)^2+3^2}\right)(t) = \frac{1}{3}\mathcal{L}^{-1}\left(\frac{3}{(s+2)^2+3^2}\right)(t)$$

From the table $= \frac{1}{3}e^{-2t}\sin 3t$.

Homework:

1. $\mathcal{L}^{-1}(F(s))(t)$, $F(s) = \frac{1}{1-s^2}$ $\quad \left(\frac{1}{1-s^2} = \frac{1}{2}\left(\frac{1}{1+s} + \frac{1}{1-s}\right)\right)$

2. $\mathcal{L}^{-1}(F(s))(t)$, $F(s) = \frac{1}{(s+3)(s-4)}$.