ODE. Lecture I.

Notations:

Defn	Definition	fcn	function
Thm	Theorem	i.e.	that is
Ex	Example	pt	point
$\Rightarrow$	therefore	elt	element
Rmk	Remark		
Pf	Proof		
HW	homework		

......

Homework will be assigned everyday after class.
and collected before the class on each M, W, F.
Questions are encouraged to ask during the class.
After every lecture, I will try my best to upload
lectures notes for that day on webpage.
The class on everyday is composed of 3 lectures. 1:00 – 1:50, 2:00 – 2:50, 3:00 – 3:50.
The rest part, please read syllabus on webpage.

As concerned with next Monday, I heard there won't
be class on that day. But I will make corresponding
adjustment about homework and exams when I make sure of the news.

(§2.1, §2.2)

Defn. An ordinary equation is an equation involving an unknown function of a single variable together with one or more of its derivatives.

Ex. $xy' + 4y = x$. ①

$y' = y^2 - t$ ②

$ty' = y$ ③

$y'' + t^2 y = \cos t$. ④

Rmk: ① ② ③ are first-order equations.

④ is second-order equations.

hence $ty''' + ty'' + y' = \sin t$.

is Third-order equations.

To solve things more general, we place differential equations into $\boxed{\text{normal form}}$

Defn. A first-order differential equation of the form

$$y' = f(t, y)$$

is said to be in normal form.

Similarly, an equation of order n has the form

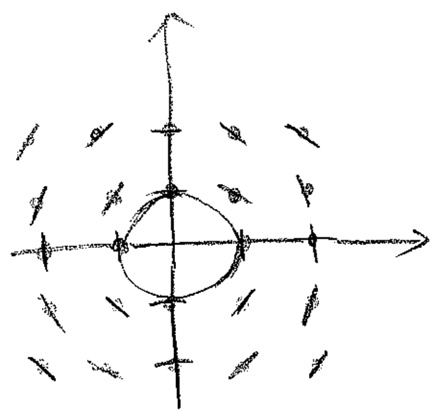
$y^{(n)} = f(t, y, y', y'', \dots, y^{(n-1)})$ is said to be in normal form.

Direction Field.

Defn. A direction field is an association of little line elts to every pt on the plane.

Rmk: We always just draw little line elts on some pts to represent it.

Like: Can first consider a lattice.

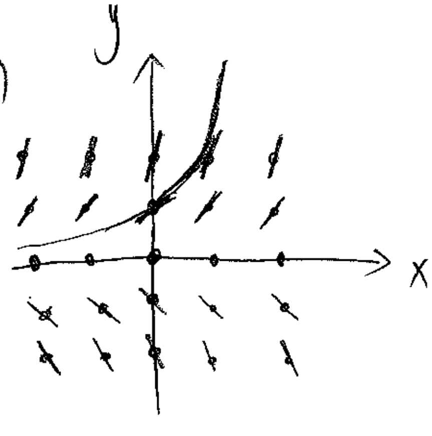


Draw little line elts at these pts.

① For $y' = f(x,y)$. how to get a direction field of $y' = f(x,y)$.

At pt (x,y), draw little line elt with slope $f(x,y)$.

Ex. $y' = y = f(x,y)$



On $(x,y) = (0,0)$, $f(x,y) = y = 0$.

On $(x,y) = (x,0)$, $f(x,y) = y = 0$.

On $(x,y) = (x,1)$, $f(x,y) = y = 1$.

On $(x,y) = (x,2)$, $f(x,y) = y = 2$.

On $(x,y) = (x,-1)$, $f(x,y) = y = -1$

On $(x,y) = (x,-2)$, $f(x,y) = y = -2$

$\boxed{\text{Integral Curve}}$

Defn. An integral curve of a direction field is a curve which has the direction of direction field as tangent vectors at every pt of this curve.

Ex. as above.

Rmk: the integral curve is the graph of solution $y=y(x)$ to $y'=f(x,y)$.

so: solving equation of $y'=f(x,y) \Longleftrightarrow$ draw integral curve of direction field of $y'=f(x,y)$.

Now we first consider :

First - Order ODE S

$$\boxed{y' = f(x, y)}$$ ← written in normal form.

Ex. $y' = \frac{x}{y}$; $y' = x - y^2$; $y' = y - x^2$.

solvable — seperable equation.

unsolvable (namely, no elementary functions can be written down as solutions)

solvable — linear equation.

Geometric View of ODE's

Analytic.		Geometric
$y' = f(x, y)$	① $\longleftrightarrow$	Direction field.
a solution $y(x)$ (it can have many solutions)	② $\longleftrightarrow$	Integral Curve.

Let me explain what's "Direction field" and "Integral Curve".

(See next page).

i.e. $y(x)$ is solution to $y' = f(x,y) \Leftrightarrow$ graph of $y(x)$ is an integral curve of direction field of $y' = f(x,y)$.

What the left side mean?

$y'(x) = f(x, y(x))$.

$\boxed{\text{Same!}}$

What the right side mean?

slope of $y(x)$ = slope of direction field at $(x, y(x))$ $\parallel$ exactly $f(x, y(x))$.

So we see they mean same thing!

Draw Direction Field.

<u>Computer Method:</u>

1. Pick (x, y).
2. $f(x, y)$ – find (calculation).
3. Draw a little line elt has slope $f(x, y)$ at (x, y).

<u>Human Method:</u>

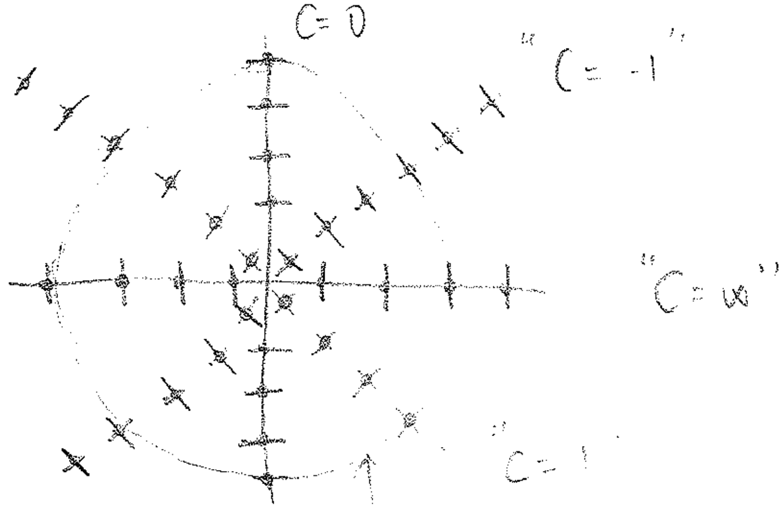
1. Pick slope C. (constant C).
2. $f(x, y) = C$. Plot the equation. Find x, y satisfy this equation, which is a curve.
3. make the curve dotted, and draw little line elts $\overbrace{\text{based}}^{\text{of slope } C}$ on pts of dotted curve.

Ex. $C = 0$

(18 min)

Ex. $y' = -\frac{x}{y}$.

so $\frac{-x}{y} = C$. $\Rightarrow$ $\boxed{y = -\frac{1}{C}x}$



integral curve.

are circles : $x^2 + y^2 = C_1$

Initial Value Problem.

Defn. A first-order differential equation together with an initial condition,

$$y' = f(x, y), \quad y(t_0) = y_0.$$

is called an initial value problem.

Ex. $y' = y^2 - t$, $y(4) = 0$.

Interval of Existence.

X

Defn. The interval of existence of a solution to a differential equation is defined to be the largest interval over which the solution can be defined and remain a solution.

Solutions to Seperable Equations.

Defn. Seperable differential Equation.

Equations of form like $\frac{dy}{dt} = g(t) f(y)$.

are called seperable differential equations.

We can solve seperable equation using the following 3 steps:

1. Seperate the variables: $\frac{dy}{f(y)} = g(t)\,dt$.
2. Integrate both sides: $\int \frac{dy}{f(y)} = \int g(t)\,dt$.
3. Solve for the equation $y(t)$, if possible.

Ex. Seperable?

$y' = e^x/(1+y)$	?	✓
$y' = \cos(xy)$	?	X
$y' = xy + y$	?	✓
$y' = e^{y/x}$	?	X

Ex : $y' = ty^2$.

Step 1 : Write the equation using dy/dt instead of y',

so $\frac{dy}{dt} = ty^2$.

Step 2 : $\frac{dy}{y^2} = t\,dt$.

Integrate it.

$$\int \frac{1}{y^2}\,dy = \int t\,dt.$$

$$\Rightarrow \quad -\frac{1}{y} = \frac{1}{2}t^2 + C.$$

Step 3 : Solve it, get the solution is

$$y(t) = -\frac{1}{\frac{1}{2}t^2 + C}$$

$$= \frac{-2}{t^2 + 2C}.$$

Ex. $x' = \frac{2tx}{1+x}$. $x(0) = 1$.

Step 1: $\frac{dx}{dt} = \frac{2tx}{1+x}$.

$\Rightarrow \frac{1+x}{x}\,dx = 2t\,dt$

$\Rightarrow (1+\frac{1}{x})\,dx = 2t\,dt$

Step 2. Integrate it, $\int (1+\frac{1}{x})\,dx = \int 2t\,dt$.

$\Rightarrow x + \ln|x| = t^2 + C$

Step 3. Solve for C.

Plug in $x(0)=1$, i.e. $t=0$, $x=1$.

$\Rightarrow 1 + \ln 1 = 0 + C$

so $1 = C$.

hence $x + \ln|x| = t^2 + 1$.

Discussion: The function $\ln|x|$ is not defined at $x=0$. so our solution can never be equal to 0. Since our initial condition is positive, and a solution must be continuous, our solution $x(t)$ must be positive for all t. So $|x| = x$ and our solution is given implicitly by $x + \ln x = t^2+1$. (*)

This is as far as we can go. We can not solve this equation. So just say the solution is defined implicitly by (*).

Ex. $y' = (\sin x)/y \qquad y(\frac{\pi}{2}) = 1.$

Step 1: $\frac{dy}{dx} = \sin x / y.$

$$\Rightarrow \quad y\,dy = \sin x\,dx.$$

Step 2: Integrate it.

$$\int y\,dy = \int \sin x\,dx$$

$$\Rightarrow \quad \frac{1}{2}y^2 = -\cos x + C$$

Step 3: Solve for C.

Plug in $y(\frac{\pi}{2})=1$, i.e. $x = \frac{\pi}{2}$, $y=1$.

$$\Rightarrow \frac{1}{2}\cdot 1^2 = -\cos\frac{\pi}{2} + C$$

$$= 0 + C$$

$$\Rightarrow \quad C = \frac{1}{2}.$$

so $\frac{1}{2}y^2 = -\cos x + \frac{1}{2}$

hence $y^2 = -2\cos x + 1.$

$$y(x) = \pm\sqrt{-2\cos x + 1}.$$

Check: $y(\frac{\pi}{2}) = 1.$

so only $y(x) = \sqrt{-2\cos x + 1}$ satisfies $y(\frac{\pi}{2}) = 1.$

hence $y(x) = \sqrt{-2\cos x + 1}$ is the solution.

:

$$y' = y/x.$$

Step 1: Write the equation using dy/dx instead of y'.

so $$\frac{dy}{dx} = y/x.$$

Step 2: $$\frac{dy}{y} = \frac{dx}{x}$$

Integrate it.

$$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx.$$

$$\Rightarrow \quad \ln y = \ln x + C.$$

Step 3: Solve it, get the solution

$$y(x) = e^{\ln x + C}$$
$$= e^{C} x.$$

Ex 3. $$y' = (1+y^2)e^x.$$

Step 1: Write the equation using dy/dx instead of y'.

So $$\frac{dy}{dx} = (1+y^2)e^x.$$

Step 2: $$\frac{dy}{1+y^2} = e^x dx$$

Integrate it. $$\int \frac{dy}{1+y^2} = \int e^x dx$$

$$\Rightarrow \quad \arctan y = e^x + C.$$

Step 3: Solve it, get the solution $y(x) = \tan(e^x + C)$.

Solving Initial Value Problem. Need use definite integration.

Ex. $y' = e^x/(1+y)$, $y(0)=1$.

Step 1: $\dfrac{dy}{dx} = e^x/(1+y)$

$\Rightarrow (1+y)dy = e^x\,dx$

Step 2: Integrate it.

$$\int (1+y)\,dy = \int e^x\,dx$$

$$\Rightarrow \quad y + \frac{1}{2}y^2 = e^x + C.$$

Step 3: Solve C.

Plug in $y(0)=1$, i.e. $x=0$, $y=1$.

$$1 + \frac{1}{2}\cdot 1^2 = e^0 + C. \quad \Rightarrow \cancel{1} + \frac{1}{2} = \cancel{1} + C.$$

$$\Rightarrow \quad C = \frac{1}{2}.$$

Step 4: $y + \frac{1}{2}y^2 = e^x + \frac{1}{2}$.

so $2y + y^2 - 2(e^x + \frac{1}{2}) = 0$. i.e. $y^2 + 2y - (2e^x+1) = 0$.

so $y = \frac{1}{2}\left[-2 \pm \sqrt{4 + 4(2e^x+1)}\right]$

$= \frac{1}{2}\left[-2 + 2\sqrt{2e^x+2}\right]$

$= -1 + \sqrt{2e^x+2}$. $\square$

Linear Equations.

Defn. A first-order linear equation is of the form

$$x' = a(t)x + f(t).$$

If $f(t) = 0$, the equation is said to be homogeneous.

Otherwise it is inhomogeneous.

Ex. $x' = \sin t \cdot x$. homogeneous.

$x' = e^{2t}x + \cos t$. inhomogeneous.

$y' = x/y$. nonlinear.

Solution of the homogeneous equation.

$$x' = a(t)x.$$

$$\Rightarrow \frac{dx}{dt} = a(t)x.$$

$$\Rightarrow \frac{dx}{x} = a(t)dt$$

$$\Rightarrow \int \frac{dx}{x} = \int a(t)dt$$

$$\Rightarrow \ln|x| = \int a(t)dt + C.$$

Exponentiating, we get $|x| = e^{\int a(t)dt + C} = e^C e^{\int a(t)dt}$.

Replace constant e^C by A.
we get the general solution is
$$x(t) = A e^{\int a(t)\,dt}.$$

Ex. $x' = \sin(t)\,x$. $\Rightarrow a(t) = \sin(t)$

get $x(t) = A e^{\int \sin t\,dt}$

$$= A e^{-\cos t + C}$$

$$= A' e^{-\cos t}.$$

<u>Solution of the inhomogeneous equation.</u>

$$x' = a(t)x + f(t)$$

$$\Rightarrow \quad x' - a(t)x = f(t)$$

then we don't know how to do it!

Now let's try an easier example.

If $a(t) = k$, i.e.

$$x' - kx = f(t).$$

we can try to solve it.

Recall: $(e^{-kt}x)' = e^{-kt}x' - e^{-kt}\cdot kx = e^{-kt}(x' - kx)$.

We can make use of this.

muliply e^{-kt} on both sides of $x' - kx = f(t)$.

We get $e^{-kt}(x' - kx) = e^{-kt} f(t)$.

$\Rightarrow (e^{-kt}x)' = e^{-kt}f(t)$

$\Rightarrow \int (e^{-kt}x)' = \int e^{-kt}f(t)$

$\Rightarrow e^{-kt}x = \int e^{-kt} f(t)$.

hence $x(t) = e^{kt}\int e^{-kt} f(t)$. Good!

So how about $x' - a(t)x = f(t)$.

Similarly, $(e^{-\int a(t)dt}x)' = e^{-\int a(t)dt}x' - e^{-\int a(t)dt}a(t)x$

$= e^{-\int a(t)dt}(x' - a(t)x)$

so multiply $e^{-\int a(t)dt}$ (called integrating factor) on both sides of

$x' - a(t)x = f(t)$.

we get $e^{-\int a(t)dt}(x' - a(t)x) = e^{-\int a(t)dt} f(t)$.

$\Rightarrow (e^{-\int a(t)dt}x)' = e^{-\int a(t)dt} f(t)$

$\Rightarrow \int (e^{-\int a(t)dt}x)' = \int e^{-\int a(t)dt} f(t)\,dt$

so $e^{-\int a(t)dt}x = \int e^{-\int a(t)dt} f(t)\,dt$

$\Rightarrow x(t) = e^{\int a(t)dt}\int e^{-\int a(t)dt} f(t)\,dt$. $\square$

Summary of the method.

So we've found a general method of solving arbitrary linear Equations.

For $$x' = a(t)x + f(t).$$

it can be solved using following 4 steps.

1. Rewrite the equation as

$$x' - a(t)x = f(t).$$

2. Multiply by the integrating factor

$$u(t) = e^{-\int a(t)dt}.$$

so the equation becomes

$$(ux)' = u(x' - ax) = uf$$

3. Integrate this equation to obtain.

$$u(t)x(t) = \int u(t)f(t)dt + C.$$

3. Solve for $x(t)$.

Ex. $x' = x + e^{-t}$.

Step 1: Rewrite it as $x' - x = e^{-t}$. $a(t) = 1$, $f(t) = e^{-t}$

Step 2: Multiply by the integrating factor

$$u(t) = e^{-\int a(t)\,dt} = e^{-\int 1\,dt} = e^{-t}.$$

so the equation becomes

$$(e^{-t}x)' = e^{-t} \cdot e^{-t}.$$

Step 3: Integrate this equation to obtain

$$e^{-t}x(t) = \int e^{-2t}\,dt + C'$$

$$= -\frac{1}{2}e^{-2t} + C$$

Step 4: $x(t) = e^{t}\left(-\frac{1}{2}e^{-2t} + C\right)$

$$= -\frac{1}{2}e^{-t} + Ce^{t}.$$

Ex. $x' = x\cos t + \cos t$. $x(0)=2$.

Step 1: Rewrite this equation as

$$x' - \cos t\, x = \cos t$$

$$a(t) = \cos t, \quad f(t) = \cos t.$$

Step 2: Multiply by the integrating factor

$$u(t) = e^{-\int a(t)\,dt} = e^{-\int \cos t\,dt} = e^{-\sin t}$$

we get, $(e^{-\sin t} x)' = e^{-\sin t}\cos t$.

Step 3: Integrate the equation to obtain

$$e^{-\sin t} x(t) = \int e^{-\sin t} \cos t\, dt + C$$

$$= \int e^{-\sin t}\, d(\sin t) + C$$

$$= -e^{-\sin t} + C$$

Step 4: so $x(t) = e^{\sin t}(-e^{-\sin t} + C)$

$$= C e^{\sin t} - 1.$$

For C, Plug for $x(0)=2$. $\Rightarrow 2 = C \cdot e^{\sin 0} - 1 \Rightarrow 2 = C - 1 \Rightarrow C = 3$.

so $x(t) = 3e^{\sin t} - 1$. $\square$

Ex of Drawing Direction Field by Human Method:

$$y' = ty^2.$$

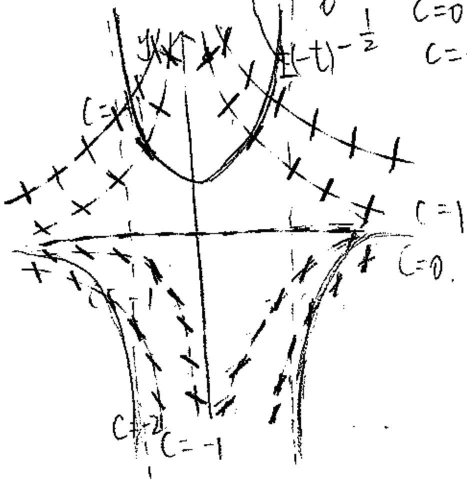
Let $y' = C$.

i.e $ty^2 = C$

so $y^2 = \frac{C}{t}$

$y = \pm\sqrt{\frac{C}{t}}$.

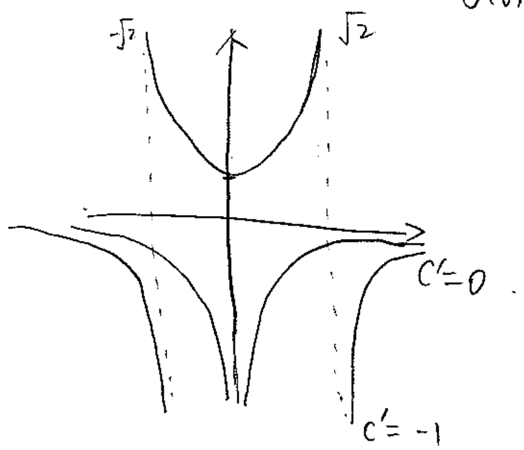
so $y = \begin{cases} \pm t^{-\frac{1}{2}}, & C=1 \\ 0 & C=0 \\ \pm(-t)^{-\frac{1}{2}} & C=-1 \end{cases}$



color chalk!

Check. $y(t) = \frac{-2}{t^2+2C'}$.

$y(t) = \begin{cases} -\frac{2}{t^2}, & C'=0 \\ -\frac{2}{t^2-2} & C'=-1. \end{cases}$



Ex. $y' = (1+y^2)$.

Ex. $y' = 2 - y$.