Lecture X

We already learn how to solve

$$y'' + py' + qy = 0.$$

or even $y''' + py'' + qy' + ry = 0$ if they have 3 distinct real eigenvalues.

Consider generalisations

One direction: Consider inhomogeneous equation

$$y'' + py' + qy = f(t)$$

and even general

$$y'' + p(t)y' + q(t)y = f(t),$$

Another direction: Consider higher-order equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = 0.$$

with the eigenvalues could be complex or with multiplicity.

Inhomogeneous Equations.

We now turn to the solution of inhomogeneous linear equations.

These are equations of the form

$$y'' + py' + qy = f(x).$$

where p, q are constants, f is a fcn.

Structure of Solutions of Inhomogeneous equations.

Thm. Suppose that y_p is a particular solution to the inhomogeneous equation, and that y_1 and y_2 form a fundamental set of solutions to be the associated homogeneous equation $y'' + py' + qy = 0$.

Then the general solution to the inhomogeneous equation is given by $y = y_p + C_1 y_1 + C_2 y_2$, where C_1, C_2 are constants.

So we only need know one particular solution to the inhomogeneous equation.

The Method of undetermined coefficients.

For $y'' + py' + qy = f(t)$.
$\uparrow$ forcing term.

Key Idea:
If the forcing term f has a form that is replicated under differentiation, then look for a solution with the same general form as the forcing term.

Case 1: Exponential forcing terms.

If $f(t) = e^{at}$, then $f'(t) = ae^{at}$.

Ex. Find a particular solution to the equation

$$y'' - y' - 2y = 2e^{-2t}.$$

So $f(t) = 2e^{-2t}$.

we look for a particular solution with the same form as f, or $y(t) = ae^{-2t}$.

here a is undetermined, which need to be set for $y(t)$ being a solution.

Since $y'(t) = -2ae^{-2t}$, $y''(t) = 4ae^{-2t}$.

hence $y'' - y' - 2y = (4a + 2a - 2a)e^{-2t} = 2e^{-2t}$

$$\Rightarrow \quad a = \frac{1}{2}.$$

hence $y(t) = \frac{1}{2}e^{-2t}$.

is a particular solution.

Ex. Find general solution to the equation

$$y'' - 3y' - 4y = 4e^{3t}$$

so $f(t) = 4e^{3t}$

We assume $y(t) = ae^{3t}$.

$\Rightarrow y'(t) = 3ae^{3t}, \quad y''(t) = 9ae^{3t}$

$\Rightarrow y'' - 3y' - 4y$

$= (9a - 9a - 4)ae^{3t} = 4e^{3t}$

$\Rightarrow a = -1$.

hence a particular solution is $y(t) = -e^{3t}$.

so Q: how is general solution?

Solve the homogeneous equation.

$$y'' - 3y' - 4y = 0.$$

The characteristic equation is $\lambda^2 - 3\lambda - 4 = 0$

$\lambda = -1$ and $\lambda = 4$.

hence $y(t) = c_1 e^{-\lambda} + c_2 e^{4\lambda}$.

hen general solution to inhomogeneous equation

$$\vec{y}(t) = -e^{3t} + c_1 e^{-t} + c_2 e^{4t}.$$

Case 2: Trigonometric forcing terms.

If $f(t) = A\cos\omega t + B\sin\omega t$.

so $f'(t) = -A\omega\sin\omega t + B\omega\cos\omega t$.

Ex. Find a particular solution to the equation

$$y'' + 2y' - 3y = 5\sin 3t + \cos 3t.$$

We look for a particular solution

$$y(t) = a\cos 3t + b\sin 3t.$$

so $y'(t) = -3a\sin 3t + 3b\cos 3t$.

$\Rightarrow y''(t) = -9a\cos 3t - 9b\sin 3t$.

so $y''(t) + 2y' - 3y$

$$= (-9a\cos 3t - 9b\sin 3t) + 2(-3a\sin 3t + 3b\cos 3t)$$

$$-3(a\cos 3t + b\sin 3t)$$

$$= (-9a + 6b - 3a)\cos 3t + (-9b - 6a - 3b)\sin 3t$$

$$= (-12a+6b)\cos 3t + (-12b-6a)\sin 3t = 5\sin 3t + \cos 3t$$

$$\Rightarrow \begin{cases} -12a+6b = 5 \\ -12b-6a = 1 \end{cases}$$

Solve (a,b). $\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\frac{11}{30} \\ \frac{1}{10} \end{pmatrix}$

so a particular solution is $\vec{y}(t) = -\frac{11}{30}\cos 3t + \frac{1}{10}\sin 3t$.

Case 3: Polynomial forcing terms.

$$f(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_{n-1} t + a_n.$$

$$\Rightarrow f'(t) = n a_0 t^{n-1} + (n-1) a_1 t^{n-2} + \cdots + a_{n-1}.$$

Ex. Find a particular solution to

$$y'' + 2y' - 3y = 3t^2 + 4t + 1$$

We look for $y(t)$ as a polynomial of same degree.

so suppose $y(t) = at^2 + bt + c$.

So $y'(t) = 2at + b$

$y''(t) = 2a$.

So $y'' + 2y' - 3y = 2a + 2(2at+b) - 3(at^2+bt+c)$

$= -3at^2 + (4a-3b)t + 2a+2b-3c$

so $\begin{cases} -3a = 3 \\ 4a - 3b = 4 \\ 2a + 2b - 3c = 1 . \end{cases}$

$\Rightarrow a = -1, \; b = -\frac{8}{3}, \quad c = -\frac{25}{9}.$

$\Rightarrow$ one particular solution is

$$y(t) = -t^2 - \frac{8}{3}t - \frac{25}{9}.$$

Exceptional Cases:

Ex. $\quad y'' - y' - 2y = 3e^{-t}.$

Choose $\quad y(t) = ae^{-t}.$

then $y' = -ae^{-t}, \quad y'' = ae^{-t}.$

so $y'' - y' - 2y = ae^{-t} + ae^{-t} - 2ae^{-t} = 0 \neq 3e^{-t}.$

so $y(t)$ could not be ae^{-t} form.

so we can assume $y(t) = a\underline{t}e^{-t}.$

so $y' = ae^{-t} - ate^{-t}, \quad y'' = -ae^{-t} - ae^{-t} + ate^{-t}.$
$\qquad\qquad = -2ae^{-t} + ate^{-t}.$

so $y'' - y' - 2y$

$= -2ae^{-t} + \cancel{ate^{-t}} - ae^{-t} + \cancel{ate^{-t}} - 2\cancel{at}e^{-t}$

$= -3ae^{-t} = 3e^{-t}$.

$\Rightarrow a = -1$,

so one particular solution is $y(t) = -te^{-t}$.

So this is like a trick.

like in some case. if ate^{-t} still doesn't work,

just continue trying with at^2e^{-t}, at^3e^{-t}, ...

<u>Combinating forcing terms.</u>

Ex. Find a particular solution to the linear equation

$$y'' - y' - 2y = 2e^{-2t} + 3e^{-t} + \cos t + (t^2+1).$$

Yes, too many terms in the forcing term.

But we know solution

$y_1(t)$ of $y'' - y' - 2y = 2e^{-2t}$

$y_2(t)$ of $y'' - y' - 2y = 3e^{-t}$

$y_3(t)$ of $y'' - y' - 2y = \cos t$

$y_4(t)$ of $y'' - y' - 2y = (t^2+1)$.

hence one particular solution $y(t) = y_1(t) + y_2(t) + y_3(t) + y_4(t)$.

Q: How about the equation

$$y'' + p(t)y' + q(t)y = g(t)?$$

Rmk: Here $p(t), q(t)$ are fcns of t.

Not as before, we restrict them to be constants.

This might see to be a great increase in generality, but there is a rather strong constraint.

Variation of Parameters

Assume that we have computed a fundamental set of solutions y_1 and y_2 to the associated homogeneous equation

$$y'' + p(t)y' + q(t)y = 0.$$

Then the general solution to the homogeneous equation is $y_h = c_1 y_1 + c_2 y_2$,

where c_1, c_2 are constants.

$\boxed{\text{Key Idea of Variation of Parameters}}$ is to replace the constants c_1 and c_2 by unknown fcns $v_1(t)$ and $v_2(t)$ and look for a particular solution to the inhomogeneous equation of the form

$$y_p = v_1 y_1 + v_2 y_2 .$$

So our task of look for y_p becomes looking for right $v_1(t)$ and $v_2(t)$, Let homogeneous equation help inhomogenous.

Ex. Find a particular solution to the equation

$$y'' + y = \tan t$$

First, $f(t) = \tan t$ is not the form which repeat itself after derivation.

Now we find a solution by the method of variation of parameters.

Step 1: Find general solutions for homogeneous equation

$$y'' + y = 0.$$

$$y_1(t) = \cos t, \quad y_2(t) = \sin t.$$

Step 2: We look for a particular solution of the form

$$y_p = v_1(t)\cos t + v_2(t)\sin t.$$

$$\Rightarrow y_p' = v_1'\cos t - v_1 \sin t + v_2' \sin t + v_2 \cos t$$

$$= (v_1'\cos t + v_2'\sin t) + v_2\cos t - v_1 \sin t.$$

Since we only need pick up one particular solution,
we can give more restrictions on v_1, v_2,
to make things easier. we can assume

$$(v_1' \cos t + v_2' \sin t) = 0.$$

(As long as we find out some v_1, v_2 satisfying all necessary conditions, our goal is achieved.)

Now $y_p' = -v_1 \sin t + v_2 \cos t.$

$$\Rightarrow y_p'' = -v_1' \sin t - v_1 \cos t + v_2' \cos t - v_2 \sin t$$

So Left side $= y_p'' + y_p$

$$= -v_1' \sin t - \cancel{v_1 \cos t} + v_2' \cos t - \cancel{v_2 \sin t}$$

$$+ \cancel{v_1 \cos t} + \cancel{v_2 \sin t} = -v_1' \sin t + v_2' \cos t.$$

Right Side $= \tan t.$

$$\Rightarrow \begin{cases} v_1' \cos t + v_2' \sin t = 0 \\ -v_1' \sin t + v_2' \cos t = \tan t. \end{cases}$$

Solve v_1', v_2'.

$$\left(\begin{array}{ccc} \cos t & \sin t & 0 \\ -\sin t & \cos t & \tan t \end{array}\right)$$

$$\xrightarrow{2 + \sin t/\cos t \ 1} \left(\begin{array}{ccc} \cos t & \sin t & 0 \\ 0 & \cos t + \frac{\sin^2 t}{\cos t} & \tan t \end{array}\right)$$

$$= \left(\begin{array}{ccc} \cos t & \sin t & 0 \\ 0 & \frac{1}{\cos t} & \tan t \end{array}\right)$$

$$\xrightarrow{2 \times \cos t} \left(\begin{array}{ccc} \cos t & \sin t & 0 \\ 0 & 1 & \sin t \end{array}\right)$$

$$\Rightarrow \quad v_1' = -\frac{\sin^2 t}{\cos t}, \quad v_2' = \sin t$$

$$\Rightarrow \quad v_1 = \int -\frac{\sin^2 t}{\cos t}\,dt, \quad v_2 = -\cos t$$

$$= -\ln|\sec t + \tan t| + \sin t.$$

$$y_p(t) = v_1 \cos t + v_2 \sin t$$

So

$$= \left(-\ln|\sec t + \tan t| + \cancel{\sin t}\right)\cos t - \cancel{\cos t \sin t}$$

$$= -\ln|\sec t + \tan t| \cdot \cos t.$$

So this is how to help solving
$$y'' + py' + qy = f(t).$$

The General Case.

Does this procedure always work?

Let's examine the general case and see what happens.

To find a particular solution to

$$y'' + p(t)y' + q(t)y = g(t),$$

We look for a solution of the form

$$y_p = v_1 y_1 + v_2 y_2 .$$

where v_1 and v_2 are fcns that are yet to be determined, and y_1 and y_2 are a fundamental set of solutions to

$$y'' + p(t)y' + q(t)y = 0.$$

We compute
$$\begin{aligned} y_p' &= v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2' \\ &= (v_1' y_1 + v_2' y_2) + v_1 y_1' + v_2 y_2' \end{aligned}$$

As in example, we assume

$$(v_1' y_1 + v_2' y_2) = 0.$$

hence $y_p' = v_1 y_1' + v_2 y_2'$

$$\Rightarrow y_p'' = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''$$

So Left Side $= y_p'' + p(t)\, y_p' + q(t)\, y_p$

$$= (v_1' y_1' + v_2' y_2' + v_1 y_1'' + v_2 y_2'')$$
$$+ p(t)(v_1 y_1' + v_2 y_2') + q(t)(v_1 y_1 + v_2 y_2)$$

$$= v_1 (y_1'' + p(t) y_1' + q(t) y_1)$$
$$+ v_2 (y_2'' + p(t) y_2' + q(t) y_2) \quad + v_1' y_1' + v_2' y_2'$$

Since y_1, y_2 are both solutions to $y'' + p(t) y' + q(t) y = 0$

$$\Rightarrow y_1'' + p(t) y_1' + q(t) y_1 = 0$$
$$y_2'' + p(t) y_2' + q(t) y_2 = 0$$

$$\Rightarrow \text{Left Side} = v_1' y_1' + v_2' y_2' = g(t) = \text{Right Side}.$$

So we get $\begin{cases} v_1' y_1 + v_2' y_2 = 0 \\ v_1' y_1' + v_2' y_2' = g(t) \end{cases}$

so to solve v_1', v_2'.

$$\begin{pmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & g(t) \end{pmatrix}$$

$$\xrightarrow{② - \frac{y_1'}{y_1} ①} \begin{pmatrix} y_1 & y_2 & 0 \\ 0 & y_2' - \frac{y_1' y_2}{y_1} & g(t) \end{pmatrix}$$

so $$v_2' = \frac{g(t)}{y_2' - \frac{y_1' y_2}{y_1}} = \frac{y_1}{y_1 y_2' - y_1' y_2} g(t).$$

$$v_1' = -\frac{y_2}{y_1} v_2' = -\frac{y_2}{y_1 y_2' - y_1' y_2} g(t).$$

so $$v_1 = \int \frac{-y_2 g(t)}{y_1 y_2' - y_1' y_2} dt$$

$$v_2 = \int \frac{y_1 g(t)}{y_1 y_2' - y_1' y_2} dt.$$

So if we know $y_1(t)$, $y_2(t)$, which are solutions to homogeneous equations, then for inhomogeneous equations we can get solutions.

Ex. Verify that $y_1(t) = t^{-1}$ and $y_2(t) = t^{-1}\ln t$ are solutions to the homogeneous equation

$$t^2 y''(t) + 3t y'(t) + y(t) = 0$$

Use variation of parameters to find the general solution to $t^2 y''(t) + 3t y'(t) + y(t) = \frac{1}{t}$.

First, verify. $y_1(t) = t^{-1}$, $y_1'(t) = -t^{-2}$, $y_1''(t) = 2t^{-3}$.

so $t^2 \cdot (2t^{-3}) + 3t(-t^{-2}) + t^{-1} = 0$.

$y_2(t) = t^{-1}\ln t$, $y_2'(t) = t^{-1} \cdot \frac{1}{t} - t^{-2}\ln t = 1 - t^{-2}\ln t$

$y_2''(t) = -t^{-2} \cdot \frac{1}{t} - 2t^{-3}\ln t = -t^{-1} - 2t^{-3}\ln t.$

so $t^2(-t^{-1}-2t^{-3}\ln t)+3t(1-t^{-2}\ln t)+t^{-1}\ln t=0.$

Second. Use variation of parameters and solutions of homogeneous equation to get solutions of inhomogeneous equations.

look for a solution

$$y_p = v_1 y_1 + v_2 y_2$$

$$= v_1 t^{-1} + v_2 t^{-1}\ln t.$$

so $y_p' = v_1\cdot(-t^{-2}) + v_1' t^{-1} + v_2(1-t^{-2}\ln t) + v_2' t^{-1}\ln t$

$= (v_1' t^{-1} + v_2' t^{-1}\ln t) + v_1(-t^{-2}) + v_2(1-t^{-2}\ln t).$

First assume $v_1' t^{-1} + v_2' t^{-1}\ln t = 0.$

so $y_p' = v_1(-t^{-2}) + v_2(1-t^{-2}\ln t).$

$\Rightarrow y_p'' = v_1'(-t^{-2}) + v_1(2t^{-3}) + v_2'(1-t^{-2}\ln t)$
$\quad + v_2(-t^{-1}-2t^{-3}\ln t)$

so $t^2 y_p'' + 3t y_p' + y_p$

$= t^2 v_1'(-t^{-2}) + t^2 v_2'(1-t^{-2}\ln t) = \frac{1}{t}.$

hence $v_1'(-t^{-2}) + v_2'(1-t^{-2}\ln t) = t^{-3}.$

so $\begin{cases} v_1' t^{-1} + v_2' t^{-1}\ln t = 0 \\ v_1'(-t^{-2}) + v_2'(1-t^{-2}\ln t) = t^{-3}. \end{cases}$

so $\begin{pmatrix} t^{-1} & t^{-1}\ln t & 0 \\ -t^{-2} & 1-t^{-2}\ln t & t^{-3} \end{pmatrix}$

$\xrightarrow{②+t^{-1}①} \begin{pmatrix} t^{-1} & t^{-1}\ln t & 0 \\ 0 & 1 & t^{-3} \end{pmatrix}$

so $v_1' = -t^{-3}\ln t$

$v_2' = t^{-3}$

so $v_1 = \int v_1'\,dt = \int -t^{-3}\ln t\,dt = \frac{1}{2}t^{-2}\ln t - \frac{1}{2}\int t^{-2}\cdot\frac{1}{t}\,dt$

$v_2 = \int t^{-3}\,dt = -\frac{1}{2}t^{-2}$. $\quad = \frac{1}{2}t^{-2}\ln t + \frac{1}{4}t^{-2}$.

so $y_p = (\frac{1}{2}t^{-2}\ln t + \frac{1}{4}t^{-2})\cdot t^{-1} + (-\frac{1}{2}t^{-2})\cdot t^{-1}\ln t$

$= \frac{1}{4}t^{-3}$. $\square$

For $y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0.$

where $a_1, \cdots, a_{n-1}, a_n$ are constants.

To reduce to 1st-order system,

introduce $x_1 = y,\ x_2 = y', \cdots, x_n = y^{(n-1)}$.

So we get
$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \vdots \\ x_n' = -a_1 x_n - a_2 x_{n-1} - \cdots - a_n x_1 \end{cases}.$$

Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$,

then
$$\vec{x}' = \underbrace{\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \\ & & & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & \cdots & -a_1 \end{pmatrix}}_{=A} \vec{x}.$$

The characteristic polynomial is

$$\det(\lambda I - A).$$

$\boxed{\text{Exercise}}$ thinking in partime:

$$\det(\lambda I - A) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n.$$

That's why we just call

$\lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$ as the characteristic polynomial.

Ex. $y'' + p y' + q y = 0.$

Introduce $x_1 = y$, $x_2 = y'$

then $\begin{cases} x_1' = x_2 \\ x_2' = -p x_2 - q x_1 \end{cases}$

so $\vec{x}' = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \vec{x}$

so $\det(\lambda I - A)$

$= \det \begin{pmatrix} \lambda & -1 \\ q & \lambda + p \end{pmatrix}$

$= \lambda(\lambda + p) + q$

$= \lambda^2 + p\lambda + q.$

Algebraic multiplicity vs Geometric multiplicity.

Suppose A is $n\times n$ matrix with real entries.

Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be a list of the distinct eigenvalues of A.

In general, the characteristic polynomial of A factors into

$$p(\lambda) = (\lambda-\lambda_1)^{q_1}(\lambda-\lambda_2)^{q_2}\cdots(\lambda-\lambda_k)^{q_k}.$$

The powers of the factors are at least 1 and satisfy

$$q_1+q_2+\cdots+q_k = n.$$

We define the algebraic multiplicity of λ_j to be q_j.

On the other hand, the geometric multiplicity of λ_j is d_j, the dimension of eigenspace of λ_j.

Ex. $A = \begin{pmatrix} 2 & \\ & 2 \end{pmatrix}$. $\lambda = 2$.

algebraic multiplicity of $\lambda=2$ is 2.

geometric multiplicity of $\lambda=2$ is 2.

Ex. $A = \begin{pmatrix} 2 & 1 \\ & 2 \end{pmatrix}$, $\lambda = 2$

algebraic multiplicity of $\lambda=2$ is 2,

geometric multiplicity of $\lambda=2$ is 1.

define

Thm. If $\lambda_1, \cdots, \lambda_k$ are distinct root to the characteristic polynomial of algebraic multiplicity $q_1, q_2, \cdots, q_k$, then

$$\begin{array}{lll} y_1(t) = e^{\lambda_1 t}, & y_2(t) = e^{\lambda_2 t}, & \cdots, \; y_k(t) = e^{\lambda_k t} \\ te^{\lambda_1 t} & te^{\lambda_2 t} & te^{\lambda_k t} \\ \vdots & \vdots & \vdots \\ t^{q_1 - 1} e^{\lambda_1 t} & t^{q_2 - 1} e^{\lambda_2 t} & t^{q_k - 1} e^{\lambda_k t}. \end{array}$$

form a fundamental set of general solutions.

Ex. Find a fundamental set of solutions to the equation

$$y'''' - 2y''' + 2y' - y = 0$$

The characteristic polynomial is

$$\lambda^4 - 2\lambda^3 + 2\lambda - 1$$

$$= (\lambda - 1)^3 (\lambda + 1).$$

Thus the roots are $1, -1$ of multiplicity $3, 1$.

so a fundamental set of solutions are

$$e^t, \; te^t, \; t^2 e^t, \; e^{-t}.$$

Ex. Find the solution of each initial-value problem

of $y''' - 6y'' + 12y' - 8y = 0$,

with $y(0) = -2$, $y'(0) = 0$, and $y''(0) = 2$.

First, the characteristic polynomial is

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8\lambda = (\lambda - 2)^3$$

hence $\lambda = 2$ with multiplicity 3.

One fundamental set of general solution is

$$\{e^{2t}, te^{2t}, t^2e^{2t}\}.$$

hence $y(t) = c_1e^{2t} + c_2te^{2t} + c_3t^2e^{2t}$.

Second, Need c_1, c_2, c_3 s.t $y(0) = -2$, $y'(0) = 0$, $y''(0) = 2$

$$\Rightarrow \begin{cases} c_1 = -2. \\ 2c_1 + c_2 = 0 \\ 2(2c_1 + c_2) + (2c_2 + 2c_3) = 2. \end{cases}$$

$$\Rightarrow c_1 = -2, \quad c_2 = 4, \quad c_3 = -3.$$

$$y'(t) = 2c_1e^{2t} + 2c_2te^{2t} + c_2e^{2t} + 2c_3t^2e^{2t} + 2c_3te^{2t}$$
$$= (2c_1 + c_2)e^{2t} + (2c_2 + 2c_3)te^{2t} + 2c_3t^2e^{2t}$$

$$y''(t) = 2(2c_1 + c_2)e^{2t} + 2(2c_2 + 2c_3)te^{2t} + (2c_2 + 2c_3)e^{2t} + 4c_3t^2e^{2t} + 4c_3te^{2t}.$$

hence $y(t) = -2e^{2t} + 4te^{2t} - 3t^2e^{2t}$.

Complex Roots.

Ex. Find a fundamental set of solutions to

$$y''' + 7y'' + 19y' + 13y = 0.$$

Firstly, the characteristic polynomial is

$$\lambda^3 + 7\lambda^2 + 19\lambda + 13$$

Can we fined the roots?

Always try $\lambda = \pm 1, 0, \pm 2$.

First $\lambda = 1$. $\times$

$\lambda = -1$. $-1+7-19+13=0$.

hence $\lambda+1$ is a factor of $\lambda^3+7\lambda^2+19\lambda+13$.

$$\begin{array}{r|l}
 & \lambda^2 + 6\lambda + 13 \\ \hline
\lambda+1 & \lambda^3+7\lambda^2+19\lambda+13 \\
 & \lambda^3+\lambda^2 \\ \hline
 & 6\lambda^2+19\lambda \\
 & 6\lambda^2+6\lambda \\ \hline
 & 13\lambda+13
\end{array}$$

so $\lambda^3+7\lambda^2+19\lambda+13 = (\lambda+1)(\lambda^2+6\lambda+13)$.

The roots are -1, $-3+2i$, $-3-2i$

So a fundamental set of <u>complex-valued solutions</u>
is $\{e^{-t}, e^{(-3+2i)t}, e^{(-3-2i)t}\}$.

for $z(t) = e^{(-3+2i)t} = e^{-3t}(\cos 2t + i\sin 2t)$
$= e^{-3t}\cos 2t + ie^{-3t}\sin 2t$.

So a fundamental set of solutions
is $\{e^{-t}, e^{-3t}\cos 2t, e^{-3t}\sin 2t\}$. $\square$

Now suppose that $\lambda = \alpha + i\beta$ is a complex root of multiplicity q, then $\bar{\lambda} = \alpha - i\beta$ is also a complex root of multiplicity q.
By the same reasoning we used in the real case, we get complex conjugate pairs of solutions
$z(t) = t^k e^{\lambda t}$ and $\bar{z}(t) = t^k e^{\bar{\lambda} t}$ where $k = 0, 1, 2, \cdots, q-1$.
To find the real solutions, we take the real and imaginary parts,
$x(t) = t^k e^{\alpha t}\cos\beta t$, $y(t) = t^k e^{\alpha t}\sin\beta t$, $k = 0, 1, 2, \cdots, q-1$.

<u>Thm.</u> If $\lambda = \alpha + i\beta$ is a complex root of the characteristic polynomial with multiplicity q, then so is $\bar{\lambda} = \alpha - i\beta$. In addition,
$x_1(t) = e^{\alpha t}\cos\beta t$, $x_2(t) = te^{\alpha t}\cos\beta t, \cdots$, and $x_q(t) = t^{q-1}e^{\alpha t}\cos\beta t$.
$y_1(t) = e^{\alpha t}\sin\beta t$, $y_2(t) = te^{\alpha t}\sin\beta t, \cdots$, and $y_q(t) = t^{q-1}e^{\alpha t}\sin\beta t$.
are $2q$ linearly independent solutions.

Ex. Find a fundamental set of solutions to

$$y'''' + 4y''' + 14y'' + 20y' + 25y = 0.$$

The characteristic polynomial is

$$\lambda^4 + 4\lambda^3 + 14\lambda^2 + 20\lambda + 25 = (\lambda^2 + 2\lambda + 5)^2$$

so we have roots $-1 \pm 2i$, each of multiplicity 2.

Thus, we have solutions

$$y_1(t) = e^{-t}\cos 2t, \quad y_2(t) = e^{-t}\sin 2t, \quad y_3(t) = te^{-t}\cos 2t,$$

$$y_4(t) = te^{-t}\sin 2t.$$