Lecture XI

Now suppose that $\lambda = \alpha + i\beta$ is a complex root of multiplicity q, then $\bar{\lambda} = \alpha - i\beta$ is also a complex root of multiplicity q. By the same reasoning we used in the real case, we get complex conjugate pairs of solutions

$$z(t) = t^k e^{\lambda t} \quad \text{and} \quad \bar{z}(t) = t^k e^{\bar{\lambda} t} \quad \text{where } k = 0, 1, 2, \cdots, q-1.$$

To find the real solutions, we take the real and imaginary parts,

$$x(t) = t^k e^{\alpha t} \cos \beta t, \quad y(t) = t^k e^{\alpha t} \sin \beta t, \quad k = 0, 1, 2, \cdots, q-1.$$

<u>Thm.</u> If $\lambda = \alpha + i\beta$ is a complex root of the characteristic polynomial with multiplicity q, then so is $\bar{\lambda} = \alpha - i\beta$. In addition,

$$x_1(t) = e^{\alpha t} \cos \beta t, \quad x_2(t) = t e^{\alpha t} \cos \beta t, \cdots, \text{ and } x_q(t) = t^{q-1} e^{\alpha t} \cos \beta t,$$

$$y_1(t) = e^{\alpha t} \sin \beta t, \quad y_2(t) = t e^{\alpha t} \sin \beta t, \cdots, \text{ and } y_q(t) = t^{q-1} e^{\alpha t} \sin \beta t.$$

are $2q$ linearly independent solutions.

Ex. Find a fundamental set of solutions to

$$y'''' + 4y''' + 14y'' + 20y' + 25y = 0.$$

The characteristic polynomial is

$$\lambda^4 + 4\lambda^3 + 14\lambda^2 + 20\lambda + 25 = (\lambda^2 + 2\lambda + 5)^2$$

so we have roots $-1 \pm 2i$, each of multiplicity 2.

Thus, we have solutions

$$y_1(t) = e^{-t}\cos 2t, \quad y_2(t) = e^{-t}\sin 2t, \quad y_3(t) = te^{-t}\cos 2t,$$

$$y_4(t) = te^{-t}\sin 2t.$$

Higher-order Linear Equations.

Defn. A linear equation of order n is of the form

$$y^{(n)} + a_1(t)\, y^{(n-1)} + \cdots + a_{n-1}(t)\, y' + a_n(t)\, y = F(t).$$

Defn. Suppose the fcns $y_1(t), y_2(t), \cdots,$ and $y_n(t)$ are all defined on the interval (α, β). The fcns are $\boxed{\text{linearly dependent}}$ if there are constants $c_1, c_2, \cdots,$ and c_n, not all equal to 0, such that $c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) = 0$, for all $t \in (\alpha, \beta)$.

Ex. Show that the fcns $\sin t$ and $\cos t$ are linearly independent.

Suppose $c_1 \sin t + c_2 \cos t = 0$.

so $\begin{cases} c_1 \sin 0 + c_2 \cos 0 = 0 & \Rightarrow c_2 = 0 \\ c_1 \sin \frac{\pi}{2} + c_2 \cos \frac{\pi}{2} = 0 & \Rightarrow c_1 = 0. \end{cases}$

$\Rightarrow$ they are linearly independent.

Q: For given set of solutions $y_1(t), y_2(t), \cdots, y_n(t)$ to a linear equation of order n, how to check whether it is a fundamental set of solutions?

Answer : Wronskian.

If you remember, for given system $\vec{y}' = A\vec{y}$,

to check $\vec{y}_1(t), \vec{y}_2(t), \cdots, \vec{y}_n(t)$ linear independent,

use $W(t) = \det(\vec{y}_1(t), \vec{y}_2(t), \cdots, \vec{y}_n(t))$.

$W(t) \not\equiv 0 \iff \vec{y}_1, \cdots, \vec{y}_n(t)$ are linear independent.

So now for fcns

$y_1(t), y_2(t), \cdots, y_n(t)$,

how to get determinant? Where is the matrix.

$$W(t) = \det \begin{pmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{pmatrix}.$$

The exponential of a matrix.

Recall: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

how about for a matrix?

$$e^A =$$

Defn. The exponential of the matrix A is defined to be

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$

Ex. Show that $e^A = \begin{pmatrix} e^{r_1} & 0 \\ 0 & e^{r_2} \end{pmatrix}$, if $A = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$.

Since $A^2 = A \cdot A = \begin{pmatrix} r_1^2 & \\ & r_2^2 \end{pmatrix}$.

$A^3 = A^2 \cdot A = \begin{pmatrix} r_1^3 & \\ & r_2^3 \end{pmatrix}$

$\cdots$

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} r_1^2 & 0 \\ 0 & r_2^2 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} r_1^3 & 0 \\ 0 & r_2^3 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1 + r_1 + \frac{r_1^2}{2!} + \frac{r_1^3}{3!} + \cdots & 0 \\ 0 & 1 + r_2 + \frac{r_2^2}{2!} + \frac{r_2^3}{3!} + \cdots \end{pmatrix} = \begin{pmatrix} e^{r_1} & 0 \\ 0 & e^{r_2} \end{pmatrix}.$$

Ex. $A = \begin{pmatrix} r_1 & & & \\ & r_2 & & \\ & & \ddots & \\ & & & r_n \end{pmatrix}$.

$$e^A = \begin{pmatrix} e^{r_1} & & & \\ & e^{r_2} & & \\ & & \ddots & \\ & & & e^{r_n} \end{pmatrix}.$$

Solution to the initial value problem.

Thm. $\vec{x}(t) = e^{tA}\vec{v}$ is the solution to the initial value problem

$$(\vec{x})' = A\vec{x} \quad \text{with } \vec{x}(0) = \vec{v}.$$

Pf: Left side $= (e^{tA}\vec{v})' = (e^{tA})'\vec{v}$

$$(e^{tA})' = \left(I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots\right)'$$

$$= A + tA^2 + \frac{t^2}{2!}A^3 + \cdots$$

$$= A\left(I + tA + \frac{t^2}{2!}A^2 + \cdots\right)$$

$$= A \cdot e^{tA}.$$

So Left side $= A \cdot e^{tA} \cdot \vec{v}$.

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Right side $= A \cdot (e^{tA}\vec{v})$.

And Plug $t=0$ into $\vec{x}(t) = e^{tA}\vec{v}$.

get $\vec{x}(0) = \vec{v}$ initial condition.

$\square$.

More properties of exponents.

Prop. 1. $e^{A+B} = e^A \cdot e^B$ iff $AB=BA$

2. $e^A \cdot e^{-A} = I$.

3. $e^{PAP^{-1}} = Pe^A P^{-1}$.

Why the proposition help us calculating $e^{tA}\vec{v}$.

$$e^{tA}\vec{v} = e^{(\lambda t I + t(A-\lambda I))}\vec{v}$$

$$= e^{\lambda t I} \cdot e^{t(A-\lambda I)}\vec{v}$$

$$= e^{\lambda t} \cdot \underbrace{e^{t(A-\lambda I)}\vec{v}}_{\substack{\text{we can calculate this} \\ \text{if } (A-\lambda I)^k \vec{v} = 0}}.$$

Prop. 1. If $[A-\lambda I]\vec{v} = 0$, then $e^{tA}\vec{v} = e^{\lambda t}\vec{v}$ for all t.

2. If $[A-\lambda I]^2 \vec{v} = 0$, then $e^{tA}\vec{v} = e^{\lambda t}(\vec{v} + t(A-\lambda I)\vec{v})$.

3. If $[A-\lambda I]^k \vec{v} = 0$, then

$$e^{tA}\vec{v} = e^{\lambda t}\left(\vec{v} + t(A-\lambda I)\vec{v} + \cdots + \frac{t^{k-1}}{(k-1)!}(A-\lambda I)^{k-1}\vec{v}\right).$$

Ex. Consider

$$A = \begin{pmatrix} -4 & -2 & 1 \\ 4 & 2 & -2 \\ 8 & 4 & 0 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \text{ and } \vec{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Compute $e^{tA}\vec{v}$ and $e^{tA}\vec{w}$.

We compute that

$$A\vec{v} = \begin{pmatrix} -4 & -2 & 1 \\ 4 & 2 & -2 \\ 8 & 4 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

so $e^{tA}\vec{v} = v$.

$$A\vec{w} = \begin{pmatrix} -4 & -2 & 1 \\ 4 & 2 & -2 \\ 8 & 4 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = -v.$$

so $A^2\vec{w} = 0$.

hence
$$\begin{aligned} e^{tA}\vec{w} &= \vec{w} + tA\vec{w} \\ &= \vec{w} - t\vec{v} \\ &= \begin{pmatrix} t \\ -2t \\ 1 \end{pmatrix}. \end{aligned}$$

Q: What are general solutions for $\vec{x}' = A\vec{x}$.

Yes: $e^{tA}\vec{u}$. for all $\vec{u}$.

Q: Can you give me a fundament set for $\vec{x}' = A\vec{x}$?

Yes: $\{e^{tA}\vec{e}_1, \dots, e^{tA}\vec{e}_n\}$.

or $\{e^{tA}\vec{v}_1, \dots, e^{tA}\vec{v}_n\}$ for any basis $\{\vec{v}_1, \dots, \vec{v}_n\}$.

This is very explicit solution.

But there is a problem.

What's $e^{tA}\vec{v}$?

It's a sum of infinitely many terms.

We cannot calculate it easily.

If there is a possibility such that

$e^{tA}\vec{v}$ is a sum of finite terms,

that's calculatable.

Prop. Suppose A is an $n\times n$ matrix and $\vec{v}$ is an n-vector.

1. If $A\vec{v}=0$, then $e^{tA}\vec{v}=\vec{v}$ for all t.
2. If $A^2\vec{v}=0$, then $e^{tA}\vec{v}=\vec{v}+tA\vec{v}$ for all t.
3. If $A^k\vec{v}=0$, then

$$e^{tA}\vec{v}=\vec{v}+tA\vec{v}+\cdots+\frac{t^{k-1}}{(k-1)!}A^{k-1}\vec{v} \quad \text{for all } t.$$

Generalized eigenvectors and the corresponding solutions.

Ex. Find a fundamental set of solutions for

$$\vec{x}' = A\vec{x}, \quad \text{where } A = \begin{pmatrix} -1 & 2 & 1 \\ 0 & -1 & 0 \\ -1 & -3 & -3 \end{pmatrix}.$$

The characteristic equation is

$$\lambda^3 - 3\lambda - 2 = (\lambda+1)(\lambda+2)^2.$$

$\lambda_1 = -1$ has multiplicity 1, and $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ is an eigenvector.

$\lambda_2 = -2$ has algebraic multiplicity 2.

geometric multiplicity 1.

Its eigenspace is generated by $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

So $\vec{x}_1(t) = e^{tA}\vec{v}_1 = e^{-t}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$,

$\vec{x}_2(t) = e^{tA}\vec{v}_2 = e^{-2t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

We need a third solution, and since $\lambda_2 = -2$ has algebraic multiplicity 2, we try to find it using part (2) of Prop (2).

(If we can find a vector $\vec{v}$ s.t.

$$[A - \lambda I]^2 \vec{v} = 0, \text{ then}$$

$$e^{tA}\vec{v} = e^{\lambda A}(\vec{v} + t(A-\lambda I)\vec{v}.)$$

We compute that $[A+2I]^2 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ -1 & -3 & -1 \end{pmatrix}^2 = \begin{pmatrix} 0 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & -8 & 0 \end{pmatrix}$.

The nullspace of this matrix has dim 2,

and it has a basis $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\end{pmatrix}$.

however, we've already computed the solution x_2 using $\vec{v}_2 = \begin{pmatrix}1\\0\\-1\end{pmatrix}$.

Notice that $\vec{v}_2$ is in the nullspace of $[A+2I]^2$.

since $[A+2I]v_2 = 0$.

So let's pick $v_3 = \begin{pmatrix}1\\0\\0\end{pmatrix}$,

Our third solution is

$$\vec{x}_3(t) = e^{tA}\vec{v}_3 = e^{-2t}\left(v_3 + t[A+2I]v_3\right)$$

$$= e^{-2t}\left(\begin{pmatrix}1\\0\\0\end{pmatrix} + t\begin{pmatrix}1 & 2 & 1\\0 & 1 & 0\\-1 & -3 & -1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}\right)$$

$$= e^{-2t}\left(\begin{pmatrix}1\\0\\0\end{pmatrix} + t\begin{pmatrix}1\\0\\-1\end{pmatrix}\right).$$

Since $x_1(0) = v_1$, $x_2(0) = v_2$, $x_3(0) = v_3$ are linearly independent,

we have a fundamental set of solutions.

Rmk: When eigenvalues have algebraic multiplicity >1,

we can compute extra solutions by looking for vectors in

the nullspace of $[A-\lambda I]^p$ for $p>1$.

If λ is an eigenvector of A, and $[A-\lambda I]^p\vec{v} = 0$ for some

integer $p \geq 1$, we will call $\vec{v}$ a <u>generalized eigenvector</u>.

In fact, generalized eigenvectors provide all of the solutions we need because of the following result from linear algebra.

<u>Thm</u>. Suppose λ is an eigenvector of A with algebraic multiplicity q. Then there is an integer $p \le q$ s.t $\dim \mathrm{Null}((A-\lambda I)^p) = q$.

The solution Procedure.

We now have a general procedure for solving linear systems. Suppose the matrix A has distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$, and that λ_j has algebraic multiplicity q_j. Since $q_1 + q_2 + \cdots + q_k = n$, we need to find q_j linear independent solutions corresponding to the eigenvalue λ_j.

To find q linearly independent solutions corresponding to an eigenvalue λ of algebraic multiplicity q:

Step 1: Find the smallest integer p s.t $\dim(\mathrm{Null}((A-\lambda I)^p)) = q$.

Step 2: Find a basis $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_q\}$ of $\mathrm{Null}((A-\lambda I)^p)$.

Step 3: For each $\vec{v}_j$, $1 \le j \le q$, we have the solution

$$x_j(t) = e^{tA}\vec{v}_j = e^{\lambda t}\left(\vec{v}_j + t[A-\lambda I]\vec{v}_j + \cdots + \frac{t^{p-1}}{(p-1)!}[A-\lambda I]^{p-1}\vec{v}_j\right).$$

Ex. $\vec{y}' = A\vec{y}$, where $A = \begin{pmatrix} -1 & -2 & 1 \\ 0 & -4 & 3 \\ 0 & -6 & 5 \end{pmatrix}$.

The characteristic polynomial is

$$\det\begin{pmatrix} \lambda+1 & 2 & -1 \\ 0 & \lambda+4 & -3 \\ 0 & 6 & \lambda-5 \end{pmatrix} = \lambda^3 - 3\lambda - 2$$

$$= (\lambda+1)^2(\lambda-2).$$

So the eigenvalues are $\underset{\substack{\downarrow \\ \text{multiplity } 2}}{-1}, 2$.

The eigenspace for $\lambda=2$ is spanned by $\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

Hence $\vec{y}_1(t) = e^{tA}\vec{v}_1 = e^{2t}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

For the eigenvalue $\lambda = -1$, we have

$$A - \lambda I = A + I = \begin{pmatrix} 0 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & -6 & 6 \end{pmatrix} \text{ and } (A+I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -9 & 9 \\ 0 & -18 & 18 \end{pmatrix}.$$

$\dim \mathrm{Null}(A+I) = 1$ $\qquad$ $\dim(\mathrm{Null}((A+I)^2)) = 2$.

means geometric multiplicity 1.

So we need to compute $e^{tA}\vec{u}$ for a basis of $\mathrm{Null}((A+I)^2)$.

It's usually a good idea to choose an eigenvector as part of our basis of $\mathrm{Null}((A+I)^2)$, since the solution corresponding to an eigenvector is easier to compute.

Hence we select $v_2 \in \text{Null}(A+I)$

$$v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

so $\vec{y}_2(t) = e^{tA}\vec{v}_2 = e^{-t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

We need our third vector $\vec{v}_3$ to be in $\text{Null}((A+I)^2)$,

but linearly independent of $\vec{v}_2$.

Perhaps, the simplest example is $v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

$$\text{so } \vec{y}_3(t) = e^{tA}\vec{v}_3$$
$$= e^{-t}\left(\vec{v}_3 + t(A+I)\vec{v}_3\right)$$
$$= e^{-t}\left(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}\right)$$
$$= e^{-t}\begin{pmatrix} -t \\ 1 \\ 1 \end{pmatrix}.$$

Ex. $\vec{y}' = Ay$, where

$$A = \begin{pmatrix} 7 & 5 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 12 & 10 & -5 & 4 \\ -4 & -4 & 2 & -1 \end{pmatrix}$$

Using a computer, we find that the characteristic polynomial is $\lambda^4 - 2\lambda^3 + 2\lambda - 1 = (\lambda+1)(\lambda-1)^3$.

So eigenvalues are -1 and 1 $\xrightarrow{}$ algebraic multiplicity 3.

For $\lambda = -1$, the eigenvector $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix}$.

Again, a computer tells us

$\text{Null}(A - I)$ has dim 2.

so geometric multiplicity of $\lambda = 1$ is 2.

And $\text{Null}(A - I)$ is spanned by

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix}.$$

Since $\vec{v}_2$ and $\vec{v}_3$ are eigenvectors, the corresponding solutions are easily computed.

They are $\vec{y}_2(t) = e^{tA}\vec{v}_2 = e^t \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$,

$$\vec{y}_3(t) = e^{tA}\vec{v}_3 = e^t \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix}.$$

Now $\text{Null}((A - I)^2)$ has dim 3.

Hence, we can find a third solution associated to $\lambda = 1$ by finding a vector $\vec{v}_4$ in $\text{Null}((A - I)^2)$ which is linearly independent of $\vec{v}_2$ and $\vec{v}_3$.

Choose $v_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$, because it has lots of zero entries.

And we can check $\vec{v}_2$, $\vec{v}_3$ and $\vec{v}_4$ are linearly independent of $\vec{v}_2$ and $\vec{v}_3$.

Since $(A-I)^2 \vec{v}_4 = 0$, the corresponding solution

is $\vec{y}_4(t) = e^{tA} \vec{v}_4 = e^t (\vec{v}_4 + t(A-I)\vec{v}_4)$

$$= e^t \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ -2 \\ 0 \end{pmatrix} \right\} = e^t \begin{pmatrix} -t \\ 0 \\ 1-2t \\ 1 \end{pmatrix}. \quad \square.$$

Ex.

For General 1st-order system.

Existence. $\vec{x}' = f(t, \vec{x})$.

Thm. Suppose the fcn $f(t, \vec{x})$ is defined and continuous in the region R and that the first partial derivatives of f are also continuous in R. Then given any pt $(t_0, x_0) \in R$, the initial value problem

$$\vec{x}' = f(t, \vec{x}) \text{ with } \vec{x}(t_0) = \vec{x}_0.$$

has a unique solution defined in an interval containing t_0.

Furthermore, the solution will be defined at least until the solution curve $t \longmapsto (t, \vec{x}(t))$ leaves the region R.

Uniqueness in the phase plane. For $\vec{x}' = f(\vec{x}) \leftarrow$ autonomous system.

Thm. Two solution curves in phase space for an autonomous system cannot meet at a pt unless the curves coincide, for $f(\vec{x})$ is continuous and has continuous derivatives.

Ex. Consider the system

$$x' = y - x(x^2 + y^2 - 1)$$
$$y' = -x - y(x^2 + y^2 - 1).$$

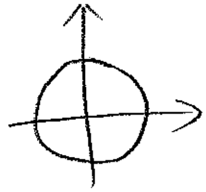
(a) Show that $x(t) = \sin t$, $y(t) = \cos t$ is a solution.

(b) Plot it in xy-plane.

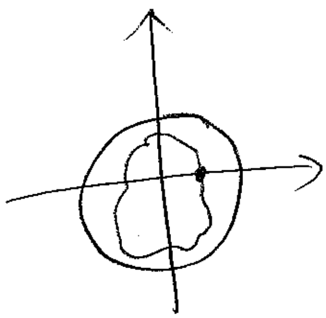
(c) Consider the solution with initial condition $x(0) = 0.5$, $y(0) = 0$. Show that $x^2(t) + y^2(t) < 1$ for all t.

(a) $\cos' t = -\sin t$.

$\sin' t = \cos t$.

(b). 

$x^2 + y^2 = 1$.

(c) 

Ex. Consider the system

$$x' = 1 - (y - \sin x)\cos x$$
$$y' = \cos x - y + \sin x.$$

(a) Show that $x(t) = t$, $y(t) = \sin t$ is a solution.

(b) Plot the solution found in part (a) in xy-plane.

(c) The solution $(x(t), y(t))$ of IVP with $x(0) = \frac{\pi}{2}$, $y(0) = 0$,

Show $y(t) < \sin x(t)$ for all t.